

QUANTITATIVE HARRIS TYPE THEOREMS FOR DIFFUSIONS AND MCKEAN-VLASOV PROCESSES

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ABSTRACT. We consider \mathbb{R}^d -valued diffusion processes of type

$$dX_t = b(X_t)dt + dB_t.$$

Assuming a geometric drift condition, we establish contractions of the transitions kernels in Kantorovich (L^1 Wasserstein) distances with explicit constants. Our results are in the spirit of Hairer and Mattingly's extension of Harris' Theorem. In particular, they do not rely on a small set condition. Instead we combine Lyapunov functions with reflection coupling and concave distance functions. We retrieve constants that are explicit in parameters which can be computed with little effort from one-sided Lipschitz conditions for the drift coefficient and the growth of a chosen Lyapunov function. Consequences include exponential convergence in weighted total variation norms, gradient bounds, bounds for ergodic averages, and Kantorovich contractions for nonlinear McKean-Vlasov diffusions in the case of sufficiently weak but not necessarily bounded nonlinearities. We also establish quantitative bounds for sub-geometric ergodicity assuming a sub-geometric drift condition.

0. INTRODUCTION

We consider \mathbb{R}^d valued diffusion processes of type

$$(1) \quad dX_t = b(X_t)dt + dB_t,$$

where $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is locally Lipschitz, and (B_t) is a d -dimensional Brownian motion. We assume non-explosiveness, and we denote the transition function of the corresponding Markov process by (p_t) .

The classical Harris' Theorem [30, 44] gives simple conditions for geometric ergodicity of Markov processes. In the case of diffusion processes on \mathbb{R}^d it goes back to Khasminskii [31, 36], in the general case it has been developed systematically by Meyn and Tweedie [45, 46, 44]. For solutions of (1), it is often not difficult to verify the assumptions in Harris' Theorem, a minorization condition for the transition probabilities on a bounded set, and a global Lyapunov type drift condition. However, it is not at all easy to quantify the constants in Harris' Theorem, and, even worse, the resulting bounds are far from sharp, and they usually have a very bad dimensional dependence. Therefore, although Harris' Theorem has become a

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standard tool in many application areas, it is mostly used in a purely qualitative way, a noteworthy exception being Roberts and Rosenthal [51].

Besides the Harris' approach, there is a standard approach for studying mixing properties of Markov processes based on spectral gaps, logarithmic Sobolev inequalities, and more general functional inequalities, see for example the monograph [3]. This approach has the advantage of providing sharp bounds in simple model cases but it sometimes yields slightly weaker, and less probabilistically intuitive results. Recent attempts [2, 1] to connect these functional inequalities to Lyapunov conditions have proven successful but they are clearly restricted to the reversible setting (or the explicit knowledge of the invariant measure). The concept of the intrinsic curvature of a diffusion process in the sense of Bakry-Emery leads to sharp bounds and many powerful results in the case where there is a strictly positive lower curvature bound κ [58]. In our context, this means that $\partial b(x) \leq -\kappa I_d$ for all x in the sense of quadratic forms.

Several of the bounds in the positive curvature case can be derived in an elegant probabilistic way by considering synchronous couplings and contraction properties in L^2 Wasserstein distances. In general, Wasserstein distances have proved crucial in the study of linear and nonlinear diffusions both via coupling techniques [13, 40, 10], or via analytic techniques based on profound results on optimal transportation, see [8, 9, 57, 4, 5] and references therein. In the case where the curvature is only strictly positive outside of a compact set, reflection coupling has been applied successfully to obtain total variation bounds for the distance to equilibrium [38] as well as explicit contraction rates of the transition semigroup in Kantorovich distances [19, 20].

An important question is how to apply a Harris' type approach in order to obtain explicit bounds that are close to sharp in certain contexts. A breakthrough towards the applicability to high- and infinite dimensional models has been made by Hairer and Mattingly in [28], and in the subsequent publications [23, 29]. The key idea was to replace the underlying couplings with finite coupling time by asymptotic couplings where the coupled processes only approach each other as $t \rightarrow \infty$ [25, 42], and the minorization condition by a contraction in an appropriately chosen Kantorovich distance. In recent years, the resulting weak Harris' theorem has been applied successfully to prove (sub)geometric ergodicity in infinite dimensional models (see e.g. [7]), and to quantify the dimension dependence in high dimensional problems [24]. Nevertheless, in contrast to the approach based on functional inequalities, the constants in applications of the weak Harris theorem are usually far from optimal. This is in particular due to the fact that the corresponding Kantorovich distance is still chosen in a somehow ad hoc way.

It turns out that a key for making the bounds more quantitative is to adapt the underlying metric on \mathbb{R}^d and the corresponding Kantorovich metric on the space of probability measures in a very specific way to the problem under consideration. For diffusion processes solving (1), this approach has been discussed in [20] assuming strict contractivity for the corresponding deterministic dynamics outside a ball. Our goal here is to replace this "Contractivity at infinity" condition by a Lyapunov condition, thus providing a more specific quantitative version of the weak Harris'

theorem. Indeed, we will define explicit metrics on \mathbb{R}^d depending both on the drift coefficient b and the Lyapunov function V such that the transition semigroup is contractive with an explicit contraction rate c w.r.t. the corresponding Kantorovich distances. Such a contraction property has remarkable consequences including not only a non-asymptotic quantification of the distance to equilibrium, but also non-asymptotic bounds for ergodic averages, gradient bounds for the transition semigroup, stability under perturbations etc.

In Section 1, we present our main results. Section 2 contains a discussion of the presented results including more detailed comparisons to the existing literature. The couplings considered here are introduced in Section 3, and the proofs of our results are given in Section 4.

1. MAIN RESULTS

We now present our main results. Let $\langle \cdot, \cdot \rangle$ and $|\cdot|$, respectively, denote the euclidean inner product and the corresponding norm on \mathbb{R}^d . We assume a generalization of a global one-sided Lipschitz condition for b combined with a Lyapunov condition. These assumptions can be weakened - we refer to [60] for an extension of the results to a more general setup.

Assumption 1 (Generalized one-sided Lipschitz condition). *There is a continuous function $\kappa : (0, \infty) \rightarrow [0, \infty)$ such that $\int_0^1 r \kappa(r) dr < \infty$, and*

$$(2) \quad \langle x - y, b(x) - b(y) \rangle \leq \kappa(|x - y|) \cdot |x - y|^2 \quad \text{for any } x, y \in \mathbb{R}^d.$$

Notice that for constant κ , (2) is a one-sided Lipschitz condition. In particular, if $b = -\nabla U$ for some function $U \in C^2(\mathbb{R}^d)$ then the assumption with constant κ is equivalent to a global lower bound on the Hessian of U . If U is strictly convex outside a ball in \mathbb{R}^d then we can choose $\kappa(r) = 0$ in (2) for sufficiently large r . Let

$$\mathcal{L} = \frac{1}{2} \Delta + \langle b(x), \nabla \rangle$$

denote the generator of the diffusion process.

Assumption 2 (Geometric drift condition). *There is a C^2 function $V : \mathbb{R}^d \rightarrow \mathbb{R}_+$ as well as constants $C, \lambda \in (0, \infty)$ such that $V(x) \rightarrow \infty$ as $|x| \rightarrow \infty$, and*

$$(3) \quad \mathcal{L}V(x) \leq C - \lambda V(x) \quad \text{for any } x \in \mathbb{R}^d.$$

It is well-known that Assumption 2 implies the non-explosiveness of solutions for (1), see e.g. [36, 46]. The function V in (3) is called a *Lyapunov function*.

Remark 1 (Choice of Lyapunov functions). *Assume there are $\mathcal{R} > 0$, $\gamma > 0$ and $q \geq 1$ such that*

$$(4) \quad \langle b(x), x \rangle \leq -\gamma |x|^q \quad \text{for any } |x| \geq \mathcal{R}.$$

Then Lyapunov functions of the following form can be chosen depending on the values of q and γ :

- Let $\alpha > 0$. If V is a C^2 function with $V(x) = \exp(\alpha |x|^q)$ outside of a compact set, then (3) holds for arbitrary $\lambda > 0$ with a finite constant $C(\alpha, \lambda)$ provided $q > 1$ and $\alpha \in (0, 2\gamma/q)$, or $q = 1$ and $\gamma > \frac{\alpha}{2} + \frac{\lambda}{\alpha}$.
- Let $\alpha > 0$ and $p \in [1, q)$. If V is C^2 with $V(x) = \exp(\alpha |x|^p)$ outside of a compact set, then (3) holds for arbitrary $\lambda > 0$ with a finite constant $C(\alpha, \lambda)$.
- Let $q \geq 2$ and $p > 0$. If V is C^2 with $V(x) = |x|^p$ outside of a compact set, then (3) holds with a finite constant $C(\lambda, p)$ if $q > 2$ and $\lambda > 0$, or if $q = 2$ and $\lambda \in (0, p\gamma)$.

Besides the two key assumptions made above, we will need a growth assumption on the Lyapunov function, cf. Assumption 3 below for our first main result, or Assumption 4 below for our second main result.

The Kantorovich distance of two probability measures μ and ν on a metric space (S, ρ) is defined by

$$\mathcal{W}_\rho(\nu, \mu) = \inf_{\gamma \in C(\nu, \mu)} \int \rho(x, y) \gamma(dx dy)$$

where the infimum is taken over all couplings with marginals ν and μ respectively. $\mathcal{W}_\rho(\nu, \mu)$ can be interpreted as the L^1 transportation cost between the probability measures ν and μ w.r.t. the underlying cost function $\rho(x, y)$. As such, it is also well-defined if ρ is only a semimetric, i.e., a function on $S \times S$ that is symmetric and non-negative with $\rho(x, y) > 0$ for $x \neq y$ but that does not necessarily satisfy the triangle inequality. In Subsections 1.1 and 1.2 we derive contractions of the transition semigroup with respect to \mathcal{W}_ρ based on two different types of underlying cost functions ρ . The first one, called the “additive distance”, is a metric, whereas the second one, called the “multiplicative distance”, in general is only a semimetric. We then consider a variation of our approach that applies to McKean-Vlasov diffusions, cf. Subsection 1.3. Subsection 1.4 discusses replacing the geometric by a subgeometric Lyapunov condition.

1.1. Geometric ergodicity with explicit constants: First main result. We first consider an underlying distance of the form

$$(5) \quad \rho_1(x, y) := [f(|x - y|) + \epsilon V(x) + \epsilon V(y)] \cdot I_{x \neq y}$$

where f is a suitable bounded, non-decreasing and concave continuous function satisfying $f(0) = 0$, and $\epsilon > 0$ is a positive constant. The choice of a distance is partially motivated by [29] where an underlying metric of the form $(x, y) \mapsto [2 + \epsilon V(x) + \epsilon V(y)] I_{x \neq y}$ is considered in order to retrieve a Kantorovich contraction based on a small set condition. We define functions $\varphi, \Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by

$$(6) \quad \varphi(r) = \exp\left(-\frac{1}{2} \int_0^r t \kappa(t) dt\right) \quad \text{and} \quad \Phi(r) = \int_0^r \varphi(t) dt$$

with κ as in Assumption 1. For constant κ , we have $\varphi(r) = \exp(-\kappa r^2/4)$. We will choose the function f to be constant outside a finite interval $[0, R_2]$ where R_2 is

defined in (13) below. Inside the interval, the function f will satisfy

$$\frac{1}{2}\Phi(r) \leq f(r) \leq \Phi(r).$$

We consider a set S_1 which is recurrent for any Markovian coupling (X_t, Y_t) of solutions of (1):

$$(7) \quad S_1 := \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : V(x) + V(y) \leq 4C/\lambda\}.$$

The set S_1 is chosen such that for $(x, y) \in \mathbb{R}^{2d} \setminus S_1$,

$$(8) \quad \mathcal{L}V(x) + \mathcal{L}V(y) \leq -\frac{\lambda}{2} (V(x) + V(y)).$$

Here the factor $1/2$ is, to some extent, an arbitrary choice. The “diameter”

$$(9) \quad R_1 := \sup \{|x - y| : (x, y) \in S_1\}$$

of the set S_1 determines our choice of ϵ in (5) :

$$(10) \quad \epsilon^{-1} := 4C \int_0^{R_1} \varphi(r)^{-1} dr = 4C \int_0^{R_1} \exp\left(\frac{1}{2} \int_0^r t \kappa(t) dt\right) dr.$$

Notice that R_1 is always finite since $V(x) \rightarrow \infty$ as $|x| \rightarrow \infty$. An upper bound is given by

$$(11) \quad R_1 \leq 2 \sup\{|x| : V(x) \leq 4C/\lambda\}.$$

We now state our third key assumption that links κ and V :

Assumption 3 (Growth condition). *There exist a constant $\alpha > 0$ and a bounded set $S_2 \supseteq S_1$ such that for any $(x, y) \in \mathbb{R}^{2d} \setminus S_2$, we have*

$$(12) \quad V(x) + V(y) \geq \frac{4C}{\lambda} \left(1 + \alpha \int_0^{R_1} \varphi(r)^{-1} dr \Phi(|x - y|)\right).$$

Assumptions linking curvature and the Lyapunov function already appeared in [12] to prove a logarithmic Sobolev inequality in the reversible setting in the case where the curvature may explode (polynomially). Similarly to R_1 we define

$$(13) \quad R_2 := \sup \{|x - y| : (x, y) \in S_2\}.$$

Note that Φ grows at most linearly. If one chooses $\alpha^{-1} = \int_0^{R_1} \varphi(r)^{-1} dr$, then Condition (12) takes the simple form

$$V(x) + V(y) \geq \frac{4C}{\lambda} \left(1 + \int_0^{|x-y|} \exp\left(-\frac{1}{2} \int_0^r t \kappa(t) dt\right) dr\right).$$

Lemma 1. *If there exists a finite constant $\mathcal{R} \geq R_1$ such that*

$$V(x) \geq 4C\lambda^{-1} (1 + 2|x|) \quad \text{for any } x \in \mathbb{R}^d \text{ with } |x| \geq \mathcal{R},$$

then Assumption 3 is satisfied with $\alpha^{-1} = \int_0^{R_1} \varphi(r)^{-1} dr$ and

$$S_2 = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : \max\{|x|, |y|\} < \mathcal{R}\}.$$

The proofs of Lemma 1 and the subsequent results in Section 1.1 are given in Section 4.1 below. Let \mathcal{P}_V denote the set of all probability measures μ on \mathbb{R}^d such that $\int V d\mu < \infty$. We can now state our first main result:

Theorem 1 (Contraction rates for additive metric). *Suppose that Assumptions 1, 2 and 3 hold true. Then there exist a concave, bounded and non-decreasing continuous function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $f(0) = 0$ and constants $c, \epsilon \in (0, \infty)$ s.t.*

$$(14) \quad \mathcal{W}_{\rho_1}(\mu p_t, \nu p_t) \leq e^{-ct} \mathcal{W}_{\rho_1}(\mu, \nu) \quad \text{for any } \mu, \nu \in \mathcal{P}_V.$$

Here the underlying distance ρ_1 is defined by (5) with ϵ determined by (10), and $c = \min\{\beta, \alpha, \lambda\}/2$ where

$$(15) \quad \beta^{-1} := \int_0^{R_2} \Phi(r) \varphi(r)^{-1} ds = \int_0^{R_2} \int_0^s \exp\left(\frac{1}{2} \int_r^s u \kappa(u) du\right) dr ds.$$

The function f is constant for $r \geq R_2$, and

$$\frac{1}{2} \leq f'(r) \exp\left(\frac{1}{2} \int_0^r t \kappa(t) dt\right) \leq 1 \quad \text{for any } r \in (0, R_2).$$

The precise definition of the function f is given in the proof in Section 4.1.

Example 1 (Bounds under global one-sided Lipschitz condition). *Suppose that there is a constant $\kappa \geq 0$ such that for any $x, y \in \mathbb{R}^d$, we have*

$$(16) \quad \langle x - y, b(x) - b(y) \rangle \leq \kappa |x - y|^2.$$

Then we can state our result in a simplified form. Suppose that Assumption 2 holds, and there is a bounded set $S_2 \supseteq S_1$ such that for any $(x, y) \notin S_2$,

$$V(x) + V(y) \geq \frac{4C}{\lambda} \left(1 + \int_0^{|x-y|} \exp(-\kappa r^2/4) dr\right)$$

Then (14) holds with contraction rate $c = \min\{2R_2^{-2}, R_1^{-1}, \lambda\}/2$ for $\kappa = 0$, and

$$(17) \quad c = \frac{1}{2} \min \left\{ \sqrt{\frac{\kappa}{\pi}} \left(\int_0^{R_2} \exp(\kappa r^2/4) dr \right)^{-1}, \left(\int_0^{R_1} \exp(\kappa r^2/4) dr \right)^{-1}, \lambda \right\}$$

for $\kappa > 0$. Here R_1 and R_2 are defined as above, and the underlying distance ρ_1 is given by (5) with $\epsilon^{-1} = 4C \int_0^{R_1} \exp(\kappa s^2/4) ds$ and a concave, bounded and non-decreasing continuous function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying $f(0) = 0$ and

$$1/2 \leq f'(r) \exp(\kappa r^2/4) \leq 1 \quad \text{for } 0 < r < R_2.$$

Remark 2. For $\kappa = 0$ and, more generally, for $\kappa R_2^2 = O(1)$, the lower bound c for the contraction rate in the example is of the optimal order $\Omega(\min\{R_2^{-2}, \lambda\})$. In general, under the assumptions made above, the bound on the contraction rate given by (17) is of optimal order in λ , and of optimal order in R_2 up to polynomial factors, see the discussion below Lemma 1 in [20].

It is well-known (see e.g. [26]), that the local Lipschitz assumption on b and Assumption 2 imply that (p_t) has a unique invariant measure $\pi \in \mathcal{P}_V$ satisfying $\int V d\pi \leq C/\lambda$. A result from [29, Lemma 2.1] then shows that the Kantorovich contraction in Theorem 1 implies bounds for the distance of μp_t and π in a weighted total variation norm.

Corollary 1 (Exponential Convergence in Weighted Total Variation Norm). *Under the assumptions of Theorem 1, there exists a unique stationary distribution $\pi \in \mathcal{P}_V$, and*

$$(18) \quad \int_{\mathbb{R}^d} V d|\mu p_t - \pi| \leq \epsilon^{-1} \exp(-ct) \mathcal{W}_{\rho_1}(\mu, \pi) \quad \text{for any } \mu \in \mathcal{P}_V.$$

In particular, for any $\delta > 0$ and $x \in \mathbb{R}^d$, the mixing time

$$\tau(\delta, x) := \inf\{t \geq 0 : \int_{\mathbb{R}^d} V d|p_t(x, \bullet) - \pi| < \delta\}$$

is bounded from above by

$$\tau(\delta, x) \leq c^{-1} \log^+ \left[\frac{R_2 \epsilon^{-1} + V(x) + C/\lambda}{\delta} \right].$$

Remark 3 (Exponential Convergence in L^p -Wasserstein distances).

For $p \in [1, \infty)$, the standard L^p -Wasserstein distance \mathcal{W}^p can be controlled by a weighted total variation norm:

$$\mathcal{W}^p(\mu, \nu) \leq 2^{1/q} \left(\int |x|^p |\mu - \nu|(dx) \right)^{1/p},$$

where $1/q + 1/p = 1$, see e.g. [57, Theorem 6.15]. Thus if there is a constant $K > 0$ such that $|x|^p \leq K V(x)$ holds for all $x \in \mathbb{R}^d$, then Corollary 1 also implies exponential convergence in L^p -Wasserstein distance.

Following ideas from [33, 34, 20], we show that Theorem 1 can be used to control the bias and the variance of ergodic averages. Let

$$(19) \quad \|g\|_{\text{Lip}(\rho)} = \sup \{|g(x) - g(y)| / \rho(x, y) : x, y \in \mathbb{R}^d, x \neq y\}$$

denote the Lipschitz norm of a measurable function $g : \mathbb{R}^d \rightarrow \mathbb{R}$ w.r.t. a metric ρ .

Corollary 2 (Ergodic averages). *Suppose that the assumptions of Theorem 1 hold true. Then for any $x \in \mathbb{R}^d$ and $t > 0$,*

$$\left| E_x \left[\frac{1}{t} \int_0^t g(X_s) ds - \int g d\pi \right] \right| \leq \frac{1 - e^{-ct}}{ct} \|g\|_{\text{Lip}(\rho_1)} \left(R_2 + \epsilon V(x) + \epsilon \frac{C}{\lambda} \right).$$

If, moreover, the function $x \mapsto V(x)^2$ satisfies the geometric drift condition

$$(20) \quad (\mathcal{L}V^2)(x) \leq C^* - \lambda^* V(x)^2 \quad \text{for any } x \in \mathbb{R}^d$$

with constants $C^*, \lambda^* \in (0, \infty)$, then

$$\text{Var}_x \left[\frac{1}{t} \int_0^t g(X_s) ds \right] \leq \frac{3}{ct} \|g\|_{\text{Lip}(\rho_1)}^2 \left(R_2^2 + 2\epsilon^2 [C^*/\lambda^* + e^{-\lambda^* t} V(x)^2] \right).$$

1.2. Geometric ergodicity with explicit constants: Second main result.

The additive distance \mathcal{W}_{ρ_1} defined in (5) is very simple, and contractivity w.r.t. \mathcal{W}_{ρ_1} even implies bounds in weighted total variation norms. However, this distance has the disadvantage that in general $\rho_1(x, y) \not\rightarrow 0$ as $x \rightarrow y$. Therefore, a contraction w.r.t. \mathcal{W}_{ρ_1} as stated in (14) can only be expected to hold if there is a coupling (X_t, Y_t) such that $P(X_t = Y_t) \rightarrow 1$ as $t \rightarrow \infty$. In the case of non-degenerate diffusions as in (1), it is not difficult to construct such a coupling, but for generalizations to infinite dimensional or nonlinear diffusions, such couplings might not be natural, see e.g. [61] and Section 1.3 below.

Partially motivated by the weak Harris Theorem in [23], we will now replace the additive metric by a multiplicative semimetric. This will allow us to derive quantitative bounds for asymptotic couplings in the sense of [25, 23], i.e., couplings for which X_t and Y_t get arbitrarily close to each other but do not necessarily meet in finite time. To this end we consider an underlying distance-like function

$$(21) \quad \rho_2(x, y) = f(|x - y|) (1 + \epsilon V(x) + \epsilon V(y))$$

where f is a suitable, non-decreasing, bounded and concave continuous function satisfying $f(0) = 0$. Note that in general, the function ρ_2 is a semimetric but not necessarily a metric, since the triangle inequality may be violated. Nevertheless, the Lipschitz norm w.r.t. ρ_2 is still well-defined by (19). In [23, Lemma 4.14], conditions are given under which ρ_2 satisfies a weak triangle inequality, i.e., under which there is a constant $C > 0$ such that $\rho(x, z) \leq C(\rho(x, y) + \rho(y, z))$ holds for all $x, y, z \in \mathbb{R}^d$. This is for example the case if V is strictly positive and grows at most polynomially, or if $V(x) = \exp(\alpha |x|)$ for large $|x|$. In any case, ρ_2 has the desirable property that $\rho_2(x, y) \rightarrow 0$ as $x \rightarrow y$.

As before, we assume that Assumptions 1 and 2 hold true. The growth condition on the Lyapunov function in Assumption 3 is now replaced by the following condition:

Assumption 4. *The logarithm of V is Lipschitz continuous, i.e., $\sup \frac{|\nabla V|}{V} < \infty$.*

Notice that in contrast to Assumption 3, Assumption 4 does not depend on κ . The global bound on ∇V can be replaced by a local bound, see [60] for an extension. Assumption 4 is satisfied if, for example, V is strictly positive, and, outside of a compact set, $V(x) = |x|^\alpha$ or $V(x) = \exp(\alpha |x|)$ for some $\alpha > 0$.

We define a bounded non-decreasing function $Q : (0, \infty) \rightarrow [0, \infty)$ by

$$(22) \quad Q(\epsilon) := \sup \frac{|\nabla V|}{\max\{V, 1/\epsilon\}}.$$

In contrast to Section 1.1, we now allow the constant ϵ in (21) to be chosen freely inside a given range. We require that

$$(23) \quad (4C\epsilon)^{-1} \geq \int_0^{R_1} \int_0^s \exp\left(\frac{1}{2} \int_r^s u \kappa(u) du + 2Q(\epsilon)(s - r)\right) dr ds$$

with C and κ given by Assumptions 2 and 1, respectively. Notice that since Q is non-decreasing, (23) is always satisfied for ϵ sufficiently small. Further below,

we demonstrate how the freedom to choose ϵ can be exploited to optimize the resulting contraction rate in certain cases. Similarly to Section 1.1, we define functions $\varphi, \Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by

$$(24) \quad \varphi(r) = \exp \left(-\frac{1}{2} \int_0^r t \kappa(t) dt - 2Q(\epsilon)r \right) \quad \text{and} \quad \Phi(r) = \int_0^r \varphi(t) dt.$$

The function f in (21) will be chosen such that

$$\frac{1}{2}\Phi(r) \leq f(r) \leq \Phi(r) \quad \text{for } r \leq R_2, \quad \text{and} \quad f(r) = f(R_2) \quad \text{for } r \geq R_2,$$

where we define

$$(25) \quad S_1 := \{ (x, y) \in \mathbb{R}^d \times \mathbb{R}^d : V(x) + V(y) \leq 2C/\lambda \},$$

$$(26) \quad S_2 := \{ (x, y) \in \mathbb{R}^d \times \mathbb{R}^d : V(x) + V(y) \leq 4C(1 + \lambda)/\lambda \},$$

$$(27) \quad R_1 := \sup \{ |x - y| : (x, y) \in S_1 \}, \quad R_2 := \sup \{ |x - y| : (x, y) \in S_2 \}.$$

Here the sets S_1 and S_2 have been chosen such that

$$\mathcal{L}V(x) + \mathcal{L}V(y) < 0 \quad \text{for } (x, y) \notin S_1, \quad \text{and}$$

$$\epsilon \mathcal{L}V(x) + \epsilon \mathcal{L}V(y) < -\frac{\lambda}{2} \min \{1, 4C\epsilon\} (1 + \epsilon V(x) + \epsilon V(y)) \quad \text{for } (x, y) \notin S_2.$$

We now state our second main result:

Theorem 2 (Contraction rates for multiplicative semimetric). *Suppose that Assumptions 1, 2, and 4 hold true. Fix $\epsilon \in (0, \infty)$ such that (23) is satisfied. Then there exist a concave, bounded and non-decreasing continuous function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $f(0) = 0$ and a constant $c \in (0, \infty)$ such that*

$$(28) \quad \mathcal{W}_{\rho_2}(\mu p_t, \nu p_t) \leq e^{-ct} \mathcal{W}_{\rho_2}(\mu, \nu) \quad \text{for any } \mu, \nu \in \mathcal{P}_V.$$

Here $c = \min \{ \beta, \lambda, 4C\epsilon\lambda \} / 2$ where

$$\beta^{-1} = \int_0^{R_2} \Phi(r) \varphi(r)^{-1} dr = \int_0^{R_2} \int_0^s \exp \left(\frac{1}{2} \int_r^s u \kappa(u) du + 2Q(\epsilon)(s - r) \right) dr ds,$$

and the underlying distance ρ_2 is defined by (21) with a function f that is constant for $r \geq R_2$ and satisfies

$$\frac{1}{2} \leq f'(r) \exp \left(\frac{1}{2} \int_0^r u \kappa(u) du + 2Q(\epsilon)r \right) \leq 1 \quad \text{for } r \in (0, R_2).$$

The precise definition of the function f is given in the proof in Section 4.2.

In order to optimize our bounds by choosing ϵ appropriately, we replace Assumption 4 by a slightly stronger condition:

Assumption 5. $\nabla V(x)/V(x) \rightarrow 0$ as $|x| \rightarrow \infty$.

If Assumption 5 holds then $Q(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. Therefore, by choosing ϵ sufficiently small, we can ensure that the term $Q(\epsilon)(s - r)$ occurring in the exponents in (23) and in the definition of β is bounded by 1. Explicitly, we choose

$$(29) \quad \epsilon = \min \left\{ Q^{-1}(R_2^{-1}), (4Ce^2 I(R_1))^{-1} \right\},$$

where $Q^{-1}(t) := \sup\{\epsilon > 0 : Q(\epsilon) \leq t\} \in (0, \infty]$ for $t > 0$ by Assumption 5, and

$$(30) \quad I(r) := \int_0^r \int_0^s \exp\left(\frac{1}{2} \int_r^s u \kappa(u) du\right) dr ds.$$

Corollary 3 (Contraction rates for multiplicative semimetric II). *Suppose that Assumptions 1, 2, and 5 hold true. Then the assertion of Theorem 2 is satisfied with ϵ given by (29) and*

$$c \geq \frac{1}{2} \min\{e^{-2}/I(R_2), \lambda, \lambda e^{-2}/I(R_1), 4C\lambda Q^{-1}(1/R_2)\}$$

The corollary is particularly useful if $b = -\nabla U$ for a convex (but not strictly convex) function U . In this case we can choose $\kappa = 0$, and hence $I(r) = r^2/2$:

Example 2 (Convex case). *Let $b(x) = -\nabla U(x)$ for a convex function $U \in C^2(\mathbb{R}^d)$, and suppose that Assumption 2 holds with V satisfying $V(x) = |x|^p$ outside of a compact set for some $p \in [1, \infty)$. Then there is a constant $A \in (0, \infty)$ such that $Q^{-1}(t) \geq At^p$ for any $t > 0$, and hence*

$$c \geq \min\{e^{-2}R_2^{-2}, \lambda/2, \lambda e^{-2}R_1^{-2}, 2C\lambda AR_2^{-p}\}.$$

In particular, $c^{-1} = O(R_2^2)$ if $V(x) = |x|^2$ outside a compact set.

Similarly as in Corollary 2 above, the bounds in Theorem 2 can be used, among other things, to control the bias and variance of ergodic averages. Furthermore a statement as in (28) implies gradient bounds for the transition kernel:

Corollary 4 (Gradient bounds for the transition semigroup). *Suppose that the assumptions in Theorem 2 are satisfied. Then*

$$\|p_t g\|_{\text{Lip}(\rho_2)} \leq e^{-ct} \|g\|_{\text{Lip}(\rho_2)}$$

holds for any $t \geq 0$ and for any function $g : \mathbb{R}^d \rightarrow \mathbb{R}$ that is Lipschitz continuous w.r.t. ρ_2 . In particular, if $p_t g$ is differentiable at x then

$$(31) \quad |\nabla p_t g(x)| \leq \|g\|_{\text{Lip}(\rho_2)} (1 + 2\epsilon V(x)) e^{-ct}.$$

The proof is included in Section 4.2 below.

1.3. McKean-Vlasov diffusions. We now apply our approach to nonlinear diffusions on \mathbb{R}^d satisfying an SDE of type

$$(32) \quad \begin{aligned} dX_t &= b(X_t) dt + \tau \int \vartheta(X_t, y) \mu_t^x(dy) dt + dB_t, & X_0 &= x, \\ \mu_t^x &= \text{Law}(X_t). \end{aligned}$$

Here $\tau \in \mathbb{R}$ is a given constant and (B_t) is a d -dimensional Brownian motion. Under appropriate conditions on the coefficients b and ϑ , Equation (32) has a unique solution (X_t) which is a nonlinear Markov process in the sense of McKean, i.e., the future development after time t depends both on the current state X_t and on the law of X_t [54, 43]. To ensure that standard existence and uniqueness results apply, we assume that b is Lipschitz (this could probably be relaxed), and in Assumption 6 below we will also assume that ϑ is Lipschitz (this is an essential

assumption for our results below). Corresponding nonlinear SDEs arise naturally as marginal limits as $n \rightarrow \infty$ of mean field interacting particle systems

$$(33) \quad dX_t^i = b(X_t^i) dt + \frac{\tau}{n} \sum_{j=1}^n \vartheta(X_t^i, X_t^j) dt + dB_t^i, \quad i = 1, \dots, n,$$

driven by independent Brownian motions B^i .

Convergence to equilibrium, or contractivity, for the nonlinear equation and the particle system are longstanding problems. Assuming $b = -\nabla V$ and $\vartheta(x, y) = \nabla W(y) - \nabla W(x)$ with smooth potentials V and W , the convex case for the nonlinear equation was tackled by Carrillo, McCann and Villani [8, 9] using PDE techniques, and by Malrieu [40] and Cattiaux, Guillin and Malrieu [10] by coupling arguments. More recently, using direct control of the derivative of the Wasserstein distance, Bolley, Gentil and Guillin [5] have proven an exponential trend to equilibrium for small bounded and Lipschitz perturbations of the strictly convex case. In the spirit of Meyn-Tweedie's approach, and via nonlinear Markov chains, Butkovsky [6] established exponential convergence to equilibrium in the bounded perturbation case. In [20, Corollary 3.4], a contraction property for the particle system (33) has been derived for sufficiently small τ with a dimension-independent contraction rate using an approximation of a componentwise reflection coupling.

We now show that a similar strategy as in [20] can be applied directly to the nonlinear equation. We assume that the interaction coefficient $\vartheta : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a globally Lipschitz continuous function:

Assumption 6. *There exists a constant $L \in (0, \infty)$ such that*

$$(34) \quad |\vartheta(x, x') - \vartheta(y, y')| \leq L \cdot (|x - y| + |x' - y'|) \quad \text{for any } x, x', y, y' \in \mathbb{R}^d.$$

In our first theorem, we assume the contractivity at infinity condition (36) instead of a Lyapunov condition. Existence and uniqueness of solutions of the nonlinear SDE can then be proven as in [10]. In that case we can obtain contractivity w.r.t. an underlying metric of type

$$(35) \quad \rho_0(x, y) = f(|x - y|)$$

where f is an appropriately chosen concave function. Let \mathcal{W}^1 denote the standard L^1 Wasserstein distance defined w.r.t. the Euclidean metric on \mathbb{R}^d . Notice that in the upcoming theorem, we allow the function κ from Assumption 1 to take negative values. We obtain the following counterpart to Corollary 3.4 in [20]:

Theorem 3 (Contraction rates for nonlinear diffusions I). *Suppose that Assumptions 1 and 6 hold true with a function $\kappa : (0, \infty) \rightarrow \mathbb{R}$ satisfying*

$$(36) \quad \limsup_{r \rightarrow \infty} \kappa(r) < 0.$$

For $x \in \mathbb{R}^d$ let μ_t^x ($t \geq 0$) denote the marginal laws of a strong solution (X_t) of Equation (32) with initial condition $X_0 = x$. Then there exist a concave and non-decreasing continuous function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $f(0) = 0$ and constants

$c, K, A \in (0, \infty)$ such that for any $\tau \in \mathbb{R}$ and $x, y \in \mathbb{R}^d$,

$$(37) \quad \mathcal{W}_{\rho_0}(\mu_t^x, \mu_t^y) \leq \exp(|\tau| K - c) t \rho_0(x, y), \quad \text{and}$$

$$(38) \quad \mathcal{W}^1(\mu_t^x, \mu_t^y) \leq 2 A \exp(|\tau| K - c) t |x - y|.$$

The constants are explicitly given by

$$(39) \quad c^{-1} = \int_0^{R_2} \int_0^s \exp\left(\frac{1}{2} \int_r^s u \kappa^+(u) du\right) dr ds,$$

$$(40) \quad K = 4 L \exp\left(\frac{1}{2} \int_0^{R_1} s \kappa^+(s) ds\right), A = \exp\left(\frac{1}{2} \int_0^{R_1} s \kappa^+(s) ds\right),$$

where

$$(41) \quad R_1 = \inf\{R \geq 0 : \kappa(r) \leq 0 \text{ for all } r \geq R\}, \quad \text{and}$$

$$(42) \quad R_2 = \inf\{R \geq R_1 : \kappa(r) R (R - R_1) \leq -4 \text{ for all } r \geq R\}.$$

The function f is linear for $r \geq R_2$, and

$$\frac{1}{2} \leq f'(r) \exp\left(\frac{1}{2} \int_0^{r \wedge R_1} s \kappa^+(s) ds\right) \leq 1 \quad \text{for } 0 < r < R_2.$$

The precise definition of the function f is given in the proof in Section 4.3.

Our next goal is to replace (36) by the following dissipativity condition:

Assumption 7 (Drift condition). *There exist constants $D, \lambda \in (0, \infty)$ such that*

$$\langle x, b(x) \rangle \leq -\lambda |x|^2 \quad \text{for any } x \in \mathbb{R}^d \text{ with } |x| \geq D.$$

Let $V(x) := 1 + |x|^2$. Assumption 7 implies that V is a Lyapunov function for the nonlinear diffusion (32), cf. Lemma 3 below.

A major difficulty in the McKean-Vlasov case is that solutions X_t and Y_t with different starting points follow dynamics with different drifts. Therefore, it is not clear how to construct a coupling (X_t, Y_t) such that $X_t = Y_t$ for $t > T$ holds for an almost surely finite stopping time T . Using the multiplicative semimetric we are still able to retrieve a local contraction:

Theorem 4 (Contraction rates for nonlinear diffusions II). *Suppose that Assumptions 1, 6 and 7 hold true. For $x \in \mathbb{R}^d$ let μ_t^x ($t \geq 0$) denote the marginal laws of a strong solution (X_t) of Equation (32) with initial condition $X_0 = x$. Then there exist a concave, bounded and non-decreasing continuous function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $f(0) = 0$ and constants $c, \epsilon, K_0, K_1 \in (0, \infty)$ such that:*

- (i) *For any $R \in (0, \infty)$ there is $\tau_0 \in (0, \infty)$ such that for any $\tau \in \mathbb{R}$ and $x, y \in \mathbb{R}^d$ with $|\tau| \leq \tau_0$ and $|x|, |y| \leq R$,*

$$(43) \quad \mathcal{W}_{\rho_2}(\mu_t^x, \mu_t^y) \leq \exp(-c t) \rho_2(x, y), \quad \text{and}$$

$$(44) \quad \mathcal{W}^1(\mu_t^x, \mu_t^y) \leq K_0 \exp(-c t) |x - y| (1 + \epsilon V(x) + \epsilon V(y)).$$

- (ii) *There is $\tau_0 \in (0, \infty)$ such that for any $\tau \in \mathbb{R}$ with $|\tau| \leq \tau_0$ and $x, y \in \mathbb{R}^d$,*

$$(45) \quad \mathcal{W}_{\rho_2}(\mu_t^x, \mu_t^y) \leq \exp(-c t) (\rho_2(x, y) + K_1 [\epsilon V(x) + \epsilon V(y)]^2).$$

The function ρ_2 is given by (21). For the explicit definition of the function f and the constants $c, \epsilon, \tau_0, K_0, K_1$ see the proof in Section 4.3.

The assumption that τ is sufficiently small is natural, since for large τ , Equation (32) can have several distinct stationary solutions. Nevertheless, we do not claim that our bound on τ is sharp.

1.4. Subgeometric ergodicity with explicit constants. We now consider the case where the drift is not strong enough to provide a Kantorovich contraction like (14). Instead of the geometric drift condition in Assumption 2 we only assume a subgeometric drift condition as it has been used for example in [15].

Assumption 8 (Subgeometric Drift Condition). *There are a function $V \in C^2(\mathbb{R}^d)$ with $\inf_{x \in \mathbb{R}^d} V(x) > 0$, a strictly positive, increasing and concave C^1 function $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\eta(V(x)) \rightarrow \infty$ as $|x| \rightarrow \infty$, as well as a constant $C \in (0, \infty)$ such that*

$$(46) \quad \mathcal{L}V(x) \leq C - \eta(V(x)) \quad \text{for any } x \in \mathbb{R}^d.$$

The following example shows how V and η can be chosen explicitly, cf. also [15].

Example 3 (Choice of V and η). *Suppose that*

$$(47) \quad \langle b(x), x \rangle \leq -\gamma |x|^q \quad \text{for } |x| \geq R$$

holds with constants $R, \gamma \in (0, \infty)$ and $q \in (0, 1)$. Let $V \in C^2(\mathbb{R}^d)$ be a strictly positive function such that outside a compact set, $V(x) = \exp(\alpha |x|^q)$ for some $\alpha \in (0, 2\gamma/q)$, and fix $\beta \in (0, \gamma - \alpha q/2)$. Then Assumption 8 is satisfied with

$$\eta(r) = \begin{cases} \alpha^{\frac{2}{q}-1} q \beta r \log(r)^{2-\frac{2}{q}} & \text{for } r \geq e^{\frac{2}{q}-1}, \\ \alpha^{\frac{2}{q}-1} \beta \left(\frac{2}{q} - 1\right)^{1-\frac{2}{q}} \left(2e^{1-\frac{2}{q}}(q-1)r^2 + (4-3q)r\right) & \text{for } r < e^{\frac{2}{q}-1}. \end{cases}$$

From now on we assume that Assumption 1 holds true, and we define the functions φ and Φ as in (6) above. Let $R_1 := \sup \{|x - y| : (x, y) \in S_1\}$, where

$$(48) \quad S_1 := \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : \eta(V(x)) + \eta(V(y)) \leq 4C\}.$$

The set S_1 is chosen such that for $(x, y) \notin S_1$,

$$(49) \quad \mathcal{L}V(x) + \mathcal{L}V(y) \leq -(\eta(V(x)) + \eta(V(y)))/2.$$

Notice that since $\eta(V(x)) \rightarrow \infty$ as $|x| \rightarrow \infty$, R_1 is finite, and S_1 is recurrent for any Markovian coupling (X_t, Y_t) of solutions of (1). Let

$$(50) \quad \epsilon^{-1} = \max \left\{ 1, 4C \int_0^{R_1} \varphi(r)^{-1} dr \right\} = \max \left\{ 1, 4C \int_0^{R_1} e^{\frac{1}{2} \int_0^r t \kappa(t) dt} dr \right\}.$$

The following growth condition on the Lyapunov function replaces Assumption 3:

Assumption 9 (Growth condition in subgeometric case). *There exist a constant $\alpha > 0$ and a bounded set $S_2 \supseteq S_1$ such that for any $(x, y) \in \mathbb{R}^{2d} \setminus S_2$, we have*

$$(51) \quad \eta(V(x)) + \eta(V(y)) \geq 4C \left(1 + \alpha \int_0^{R_1} \varphi(r)^{-1} dr \eta(\Phi(|x - y|)) \right).$$

Notice that Φ grows at most linearly. Let $R_2 := \sup \{|x - y| : (x, y) \in S_2\}$. We state our main result for the subgeometric case.

Theorem 5 (Subgeometric decay rates). *Suppose that Assumptions 1, 8 and 9 hold true. Then there exist a concave, bounded and non-decreasing continuous function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $f(0) = 0$ and constants $c, \epsilon \in (0, \infty)$ s.t.*

$$(52) \quad \|p_t(x, \cdot) - p_t(y, \cdot)\|_{TV} \leq \frac{\rho_1(x, y)}{H^{-1}(ct)} \quad \text{for any } x, y \in \mathbb{R}^d \text{ and } t \geq 0.$$

Here the distance ρ_1 is defined by (5) and (50), the function $H : [l, \infty) \rightarrow [0, \infty)$ is given by

$$(53) \quad H(t) := \int_l^t \frac{1}{\eta(s)} ds \quad \text{with} \quad l = 2\epsilon \inf_{x \in \mathbb{R}^d} V(x),$$

$c = \min \{\alpha, \beta, \gamma\} / 2$ where β is given by (15), and

$$\gamma = \inf \{ \epsilon \eta(r) / \eta(\epsilon r) : r \geq l/\epsilon \}.$$

The function f is constant for $r \geq R_2$, and

$$\frac{1}{2} \leq f'(r) \exp \left(\frac{1}{2} \int_0^r t \kappa(t) dt \right) \leq 1 \quad \text{for any } r \in (0, R_2).$$

The precise definition of the function f is given in the proof in Section 4.4.

The crucial difference in comparison to Theorem 1 is that we do not provide upper bounds on \mathcal{W}_{ρ_1} , but use the additive metric to derive moment bounds for coupling times instead. These bounds are partially based on a technique from [27], see Section 2.6 further below.

Remark 4. Since $\eta(s)$ is concave, it is growing at most linearly as $s \rightarrow \infty$. In particular, $\int_l^\infty (1/\eta(s)) ds = \infty$, and thus the inverse function H^{-1} maps $[0, \infty)$ to $[l, \infty)$. Since $\epsilon \leq 1$ and η is increasing, we always have $\gamma \geq \epsilon$. If $\eta(r) = r^a$ for some $a \in (0, 1)$ then $\gamma \geq \epsilon^{1-a}$.

It is well-known that the local Lipschitz assumption on b together with Assumption 8 implies the existence of a unique invariant probability measure π satisfying $\int \eta(V(x)) \pi(dx) \leq C$, see e.g. [27, Section 4]. Theorem 5 can be used to quantify the speed of convergence towards the invariant measure using cut-off arguments. Following [27, Section 4], we obtain:

Corollary 5. *Under the Assumptions of Theorem 5,*

$$\|p_t(x, \cdot) - \pi\|_{TV} \leq \frac{R_2 + \epsilon V(x)}{H^{-1}(ct)} + \frac{(2\epsilon b + 1)C}{\eta(b H^{-1}(ct))} \quad \text{for any } x \in \mathbb{R}^d \text{ and } t \geq 0,$$

where $b := \eta^{-1}(2C)/l$.

The proofs are given in Section 4.4.

2. DISCUSSION

2.1. Comparison to Meyn-Tweedie approach. The classical Harris theorem, as propagated by Meyn-Tweedie, allows to derive geometric ergodicity for a large class of Markov chains under conditions which are easy to verify. The approach is very generally applicable, but it is usually not trivial to make the results quantitative. The first assumption is that the Markov chain at hand is recurrent w.r.t. some bounded subset S of the state space and that one has some kind of control over the average length of excursions from this set. The second assumption which is typically imposed is a minorization condition which often takes the following form: There are constants $t, \epsilon \in (0, \infty)$ and a probability measure Q such that

$$(54) \quad p_t(x, \cdot) \geq \epsilon Q(\cdot)$$

holds for all $x \in S$, where p_t denotes the transition kernel of the chain.

The recurrence condition can be quantified performing direct computations with the generator of the Markov chain via Lyapunov techniques. The minorization condition is usually much harder to quantify. In the context of diffusions of the form (1) there are abstract methods available which allow to conclude that the condition (54) can indeed be satisfied, cf. [37, Remark 1.29]. Nevertheless, using such methods, it is not clear how the resulting constant ϵ depends on the drift coefficient b , and how a perturbation of b translates to a change of ϵ . In the diffusion setting, Roberts and Rosenthal developed in [51] a method to provide explicit bounds for ϵ that are closely connected to the drift coefficient b . Their method is based on reflection coupling and an application of the Bachelier-Lévy formula. In comparison to their results, we establish contractions of the transition kernels, and our contraction rates are based only on *one-sided* Lipschitz bounds for the drift coefficient. This often leads to much more precise bounds.

2.2. Relation to functional inequalities. Functional inequalities are now a common tool to get rates for convergence to equilibrium in L^2 distance or in entropy. For the class of diffusion processes considered here, the Poincaré inequality takes the form

$$(55) \quad \text{Var}_\pi(f) \leq \frac{1}{2} C_P \int |\nabla f|^2 d\pi$$

for smooth functions f , where π is the stationary distribution. (55) is equivalent to L^2 convergence to equilibrium (and in fact L^2 contractivity) with rate C_P^{-1} . It turns out to be quite difficult to prove a Poincaré inequality for a general non-reversible diffusion such as (1), as usual criteria rely on the explicit knowledge of the invariant probability measure π . If we assume that $b(x) = -\nabla V(x)/2$ then the diffusion is reversible with respect to $d\pi = e^{-V} dx$ and plenty of criteria are available to prove Poincaré inequalities. In particular, it is shown in [1], that if there exists a set B , constants $\lambda, C \in (0, \infty)$, and a positive twice continuously differentiable function V such that

$$\mathcal{L}V \leq -\lambda V + C1_B,$$

and a local Poincaré inequality of the form

$$\int_B (f - \pi(f 1_B))^2 d\pi \leq \frac{1}{2} \kappa_B \int |\nabla f|^2 d\pi$$

holds, then a global Poincaré inequality holds with constant $C_P = \lambda^{-1}(1 + C\kappa_B)$. Note that a Poincaré inequality implies back the Lyapunov condition. Using the additive metric and Corollary 1, one has that a Poincaré inequality holds but the identification of the constant is a hard task in general. However, using the multiplicative metric and the gradient bounds of Corollary 4, one may prove that in the reversible case a Poincaré inequality holds with the same constant c than in Corollary 4. Here the reflection coupling serves as an alternative to a local Poincaré inequality. The latter is usually established via Holley-Stroock's perturbation argument which may lead to quite poor estimates.

Notice also that, by a result of Sturm and von Renesse [58], for a reversible diffusion with stationary distribution $e^{-V} dx$, a strict contraction in L^p Wasserstein distance is *equivalent* to a lower bound on the Hessian of V . The latter condition is a special case of the Bakry-Emery criterion and usually linked to logarithmic Sobolev inequalities. In [11], a reinforced Lyapunov condition has been used to prove stronger functional inequalities than Poincaré inequalities (namely super Poincaré inequalities, including logarithmic Sobolev inequalities). In a similar spirit, we are now able to remove the global curvature condition assuming a reinforced Lyapunov condition. Note however, that although our results are sufficient to prove back some Poincaré inequality, it does not seem possible to get stronger inequalities starting from our contractions.

2.3. Dimension dependence. In our results above, dependence on the dimension d usually enters through the value of the constant C in the Lyapunov condition, which affects the size of R_2 . For example, in Theorem 1, the contraction rate is $c = \min\{\alpha, \beta, \lambda\}/2$, where α and λ are given by Assumptions 3 and 2 respectively, and the constant β defined in (15) depends both on R_2 and on the function κ in Assumption 1. In order to illustrate the dependence on the dimension of R_2 , let us assume that there are constants $A, \gamma \in (0, \infty)$ and $q \geq 1$ such that

$$(56) \quad \langle x, b(x) \rangle \leq -\gamma |x|^q \quad \text{for all } |x| \geq A.$$

Suppose first that $q = 2$. Then $V(x) = 1 + |x|^2$ satisfies the Lyapunov condition in Assumption 2 with constants $C = O(d)$ and $\lambda = \Omega(1)$. In this case, the set S_2 in Assumption 3 can be chosen such that $R_2 = O(\sqrt{d})$. Hence assuming a one-sided Lipschitz condition with constant κ as in Example 1, the lower bound c for the contraction rate in Theorem 1 is of order $\Omega(1/d)$ if $\kappa = 0$ (convex case), or, more generally, if $\kappa = O(1/d)$. On the other hand, for $\kappa = \Omega(1)$, c is exponentially small in the dimension. By Example 2, similar statements hold true for the lower bound on the contraction rate w.r.t. the multiplicative semimetric derived in Corollary 3.

Now assume more generally $q \geq 1$. In this case, a Lyapunov function with polynomial growth does not necessarily exist. Instead, by Remark 1, one can choose a Lyapunov function V with constant $\lambda = 1$ such that outside of a compact set, $V(x) = \exp(a|x|^q)$ for some $a < 2\gamma/q$. In this case, $C = O(\exp(\eta d))$ for some

finite constant $\eta > 0$, and one can choose R_2 of order $O(d^{1/q})$. Again, assuming a one-sided Lipschitz condition, the constant c in Theorem 1 is of polynomial order $\Omega(d^{-2/q})$ if $\kappa = 0$ (convex case), or, more generally, if $\kappa = O(d^{-2/q})$. For the multiplicative semimetric, we are not able to prove a polynomial order in the dimension in this case; for $q \in (1, 2)$, an application of Corollary 3 with a Lyapunov function satisfying $V(x) = \exp(|x|^\alpha)$ for large $|x|$ for some $\alpha \in (2 - q, 1)$ at least yields a sub-exponential order in d . For $\kappa = \Omega(1)$, the values of c decay exponentially in the dimension.

We finally remark that in some situations it is possible to combine the techniques presented here with additional arguments to derive explicit and dimension-free contraction rates for diffusions, see for example [61].

2.4. Extensions of the results. Similarly as in [20], the results presented above can be easily generalized to diffusions with a constant and non-degenerate diffusion matrix σ . In the case of non-constant and non-degenerate diffusion coefficients $\sigma(x)$, it should still be possible to retrieve related results replacing reflection coupling by the *Kendall-Cranston coupling* w.r.t. the intrinsic Riemannian metric induced by the diffusion coefficients.

The main contraction results, Theorem 1 and Theorem 2, are based on Assumption 1, a *global* generalized one-sided Lipschitz condition. It is possible to relax this condition to a *local* bound which, up to some technical details, holds only on the set for which the coupling (X_t, Y_t) is recurrent. A corresponding generalization is given in [60].

In the recent work [39], Majka extends the results from [20] to stochastic differential equations driven by Lévy jump processes with rotationally invariant jump measures, thus deriving Kantorovich contractions for the transition semigroups with explicit constants, see also [59]. One of the key assumptions in [39] is the “contractivity at infinity” condition (36). Using an additive distance similar to (5), it should be possible to extend the results presented there, replacing the latter assumption by a more general geometric drift condition.

An extension of the theory presented in this paper to a class of degenerate and infinite-dimensional diffusions is considered in [61] combining asymptotic couplings with the multiplicative distance (21).

In this work, we derive explicit contraction rates for diffusion processes. An important question is whether similar results can be obtained for time-discrete approximations. There are at least two different approaches to tackle this question. The first approach, which is considered in forthcoming work by one of the authors, is to establish related coupling approaches directly for Markov chains. Another possibility is to consider time discretizations as a perturbation of the diffusion process, and to apply directly the contraction results for the diffusion, cf. [14, 17, 18] and also [50, 53, 48, 21, 52].

2.5. McKean-Vlasov equations. For the class of nonlinear diffusions considered above, Theorems 3 and 4 above considerably relax assumptions in previous works. Both the PDE approach in [8] and the approach based on synchronous coupling in [40, 10] require global positive curvature bounds. In the case where the curvature is

strictly positive with degeneracy at a finite number of points, algebraic contraction rates have been derived by synchronous coupling. The dissipation of W^2 approach in [5] yields exponential decay to equilibrium for sufficiently small τ provided the confinement and interaction forces both derive from a potential, the confinement force satisfies condition (36), and the interaction potential is bounded with a lower bound on the curvature. The approach in [6] yields exponential convergence to equilibrium in total variation distance in the small and bounded interaction case. Theorem 3 above relaxes these assumptions on the interaction potential while requiring only a “strict convexity at infinity” condition on the confinement potential. Moreover, Theorem 4 replaces the latter condition on the confinement potential by the dissipativity condition in Assumption 7. With additional technicalities, it should be possible to relax this dissipativity condition to $\langle x, b(x) \rangle \leq -\lambda |x|$.

2.6. Subgeometric ergodicity. Our results in the subgeometric case can be interpreted as a variation of statements from the lecture notes [27, Section 4]. There, M. Hairer derives subgeometric ergodicity for diffusions, estimating hitting times of recurrent sets and combining these with a minorization condition. While the principle result from [27, Section 4] is already contained in [2, 15], the method of proof shows new and interesting aspects avoiding discrete-time approximations. The main tool used is the following statement, which gives an elegant proof for the integrability of hitting times:

Lemma 2 ([27], reformulated). *Let $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a strictly positive, increasing and concave C^1 function and denote by (Z_t) a continuous semimartingale, i.e. $Z_t = Z_0 + A_t + M_t$, where (A_t) is of finite variation, (M_t) is a local martingale, $E[Z_0] < \infty$ and $A_0 = M_0 = 0$. Let T be a stopping time. If there are constants $l, c \in (0, \infty)$ such that*

$$(57) \quad Z_t \geq l \quad \text{and} \quad A_t \leq -c \int_0^t \eta(Z_s) ds \quad \text{almost surely for } t < T,$$

then T is almost surely finite and satisfies the inequality

$$E [H^{-1}(cT)] \leq E [H^{-1}(H(Z_T) + cT)] \leq E[Z_0],$$

where $H : [l, \infty) \rightarrow [0, \infty)$ is given by $H(t) := \int_l^t \frac{1}{\eta(s)} ds$.

Our result for the subgeometric case, Theorem 5, relies on the above tool. The main difference to [27] is that we do not impose any kind of minorization condition or renewal theory. Instead we consider a reflection coupling (X_t, Y_t) of the diffusions, defined in Section 3.2, and we directly establish bounds on the integrability of the coupling time $T := \inf\{t \geq 0 : X_t = Y_t\}$ using Lemma 2 and the additive distance (5). For the reader’s convenience, a proof of Lemma 2 is included in Section 4.4. It should be mentioned that subgeometric ergodicity of Markov processes has been studied by many others authors in various settings, see [16, 22, 47, 55, 56, 7, 41, 32] and the references therein.

3. COUPLINGS

3.1. Synchronuous coupling for diffusions. Given initial values $(x_0, y_0) \in \mathbb{R}^{2d}$ and a d -dimensional Brownian motion (B_t) , we define a *synchronuous coupling* of two solutions of (1) as a diffusion process (X_t, Y_t) with values in \mathbb{R}^{2d} solving

$$\begin{aligned} dX_t &= b(X_t) dt + dB_t, & X_0 &= x_0, \\ dY_t &= b(Y_t) dt + dB_t, & Y_0 &= y_0. \end{aligned}$$

3.2. Reflection coupling for diffusions. Given initial values $(x_0, y_0) \in \mathbb{R}^{2d}$ and a d -dimensional Brownian motion (B_t) , we define a *reflection coupling* of two solutions of (1) as a diffusion process (X_t, Y_t) with values in \mathbb{R}^{2d} solving

$$\begin{aligned} dX_t &= b(X_t) dt + dB_t, & (X_0, Y_0) &= (x_0, y_0), \\ dY_t &= b(Y_t) dt + (I - 2e_t \langle e_t, \cdot \rangle) dB_t & \text{for } t < T, \\ Y_t &= X_t & \text{for } t \geq T, \end{aligned}$$

where $T = \inf\{t \geq 0 : X_t = Y_t\}$ is the coupling time. For $t < T$ the unit vector e_t is given by $e_t = (X_t - Y_t)/|X_t - Y_t|$. The coupling goes back to [38].

3.3. Coupling for McKean-Vlasov processes. We construct a coupling for two solutions of (32). The coupling will be realized as a process (X_t, Y_t) with values in \mathbb{R}^{2d} . We first describe the coupling in words: We fix a parameter $\delta > 0$ and use a reflection coupling of the driving Brownian motions whenever $|X_t - Y_t| \geq \delta$. If, on the other hand, $|X_t - Y_t| \leq \delta/2$ we use a synchronuous coupling. Inbetween there is a transition region, where a mixture of both couplings is used. One should think of δ being close to zero.

The technical realization of the coupling is near to [20]. In order to implement the above coupling, we introduce Lipschitz functions $\text{rc} : \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, 1]$ and $\text{sc} : \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, 1]$ satisfying

$$(58) \quad \text{sc}^2(x, y) + \text{rc}^2(x, y) = 1.$$

We impose that $\text{rc}(x, y) = 1$ holds whenever $|x - y| \geq \delta$ and $\text{rc}(x, y) = 0$ holds if $|x - y| \leq \delta/2$. The functions rc and sc can be constructed using standard cut-off techniques. Notice that in the case where the drift coefficient b and the nonlinearity ϑ are Lipschitz, equation (32) admits a unique, strong and non-explosive solution (X_t) for any initial value $x_0 \in \mathbb{R}^d$. The uniqueness holds pathwise and in law. Moreover, the law $\mu_t^{x_0}$ of X_t has finite second moments, i.e. $\int |y|^2 \mu_t^{x_0}(dy) < \infty$, see [43, Theorem 2.2] and [54]. For a fixed $x_0 \in \mathbb{R}^d$ we define

$$b^{x_0}(t, y) = b(y) + \theta \int \vartheta(y, z) \mu_t^{x_0}(dz).$$

The results from [43, Theorem 2.2] imply that the function $b^{x_0} : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is continuous. It is easy to see that Assumption 6, in combination with a Lipschitz bound on b , implies that there is $M > 0$ such that

$$(59) \quad \sup_{t \geq 0} |b^{x_0}(t, y) - b^{x_0}(t, z)| \leq M \cdot |y - z| \quad \text{for any } y, z \in \mathbb{R}^d.$$

Fix now initial values $(x_0, y_0) \in \mathbb{R}^{2d}$, the parameter $\delta > 0$ and two independent Brownian motions (B_t^1) and (B_t^2) . For the given x_0 and y_0 we construct diffusion coefficients b^{x_0} and b^{y_0} as above and define the coupling $(U_t) = (X_t, Y_t)$ as the solution of the standard diffusion

$$\begin{aligned} dX_t &= b^{x_0}(t, X_t) dt + \text{rc}(U_t) dB_t^1 + \text{sc}(U_t) dB_t^2 \\ dY_t &= b^{y_0}(t, Y_t) dt + \text{rc}(U_t) (I - 2e_t \langle e_t, \cdot \rangle) dB_t^1 + \text{sc}(U_t) dB_t^2, \end{aligned}$$

with $(X_0, Y_0) = (x_0, y_0)$ and

$$e_t = \begin{cases} \frac{X_t - Y_t}{|X_t - Y_t|} & X_t \neq Y_t \\ u & X_t = Y_t, \end{cases}$$

where $u \in \mathbb{R}^d$ is some arbitrary fixed unit vector. Note that the concrete choice of u is irrelevant for the dynamic, since $\text{rc}(x, x) = 0$. Inequality (59) implies that the above diffusion process admits a unique, strong and non-explosive solution. Using Levy's characterization of Brownian motion and (58), one can verify that the marginal processes (X_t) and (Y_t) solve the standard equations

$$(60) \quad dX_t = b^{x_0}(t, X_t) dt + B_t, \quad X_0 = x_0$$

$$(61) \quad dY_t = b^{y_0}(t, Y_t) dt + \hat{B}_t, \quad Y_0 = y_0.$$

with respect to the Brownian motions

$$(62) \quad \begin{aligned} B_t &= \int_0^t \text{rc}(U_s) dB_s^1 + \int_0^t \text{sc}(U_s) dB_s^2 \quad \text{and} \\ \hat{B}_t &= \int_0^t \text{rc}(U_s) (I - 2e_s \langle e_s, \cdot \rangle) dB_s^1 + \int_0^t \text{sc}(U_s) dB_s^2. \end{aligned}$$

Since the solutions (X_t) and (Y_t) of (60) and (61) are pathwise unique, they coincide a.s. with the strong solutions of (32) w.r.t. the Brownian motions (B_t) and (\hat{B}_t) and initial values x_0 and y_0 , respectively. Hence (X_t, Y_t) is indeed a coupling for (32).

4. PROOFS

Let us start with a crucial tool which will be used throughout our proofs: A general construction of the function f appearing in the main theorems, characterized by a differential inequality.

We define a concave function $f : [0, \infty) \rightarrow [0, \infty)$ depending on various parameters. Fix constants $R_1, R_2 \in \mathbb{R}_+$ such that $R_1 \leq R_2$, and let functions

$$h : [0, R_2] \rightarrow [0, \infty), \quad j : [0, R_2] \rightarrow [0, \infty) \quad \text{and} \quad i : [0, R_1] \rightarrow [0, \infty)$$

be given. We suppose that i and j are continuous, that j is non-decreasing and that h is continuously differentiable with $h' \geq 0$. The function f is given by

$$f(r) = \int_0^{r \wedge R_2} \varphi(s) g(s) ds,$$

where φ and g are defined as

$$(63) \quad \varphi(r) = \exp(-h(r)) \quad \text{and}$$

$$(64) \quad g(r) = 1 - \frac{\beta}{4} \int_0^{r \wedge R_2} j(\Phi(s)) \varphi(s)^{-1} ds - \frac{\xi}{4} \int_0^{r \wedge R_1} i(s) \varphi(s)^{-1} ds.$$

Here the function Φ and the constants β and ξ are given by

$$(65) \quad \Phi(r) = \int_0^r \varphi(s) ds, \quad \beta^{-1} = \int_0^{R_2} j(\Phi(s)) \varphi(s)^{-1} ds, \quad \xi^{-1} = \int_0^{R_1} i(s) \varphi(s)^{-1} ds.$$

The function f is a generalization of the concave distance function constructed in [20]. It is continuously differentiable on $(0, R_2)$ and constant on $[R_2, \infty)$. The derivative f' on $(0, R_2)$ is given by the product φg , where φ and g are positive and non-increasing functions. Hence f is a concave and non-decreasing function. Notice that g maps the interval $[0, R_2]$ into $[1/2, 1]$, which implies that the following inequalities hold for any $r \in [0, R_2]$:

$$(66) \quad r \varphi(R_2) \leq \Phi(r) \leq 2 f(r) \leq 2 \Phi(r) \leq 2 r.$$

The crucial property of the function f is that it is twice continuously differentiable on $(0, R_1) \cup (R_1, R_2)$ and that it satisfies on this set the (in)equality

$$(67) \quad \begin{aligned} f''(r) &= -h'(r) f'(r) - \frac{\beta}{4} j(\Phi(r)) - \frac{\xi}{4} i(r) I_{r < R_1} \\ &\leq -h'(r) f'(r) - \frac{\beta}{4} j(f(r)) - \frac{\xi}{4} i(r) I_{r < R_1}. \end{aligned}$$

Observe that f is not continuously differentiable at the point R_2 and thus we sometimes work with the left-derivative f'_- which exists everywhere. The function f can formally be extended to a concave function on \mathbb{R} by setting $f(r) = -r$ for $r < 0$. We can associate with f a signed measure μ_f on \mathbb{R} , which takes the role of a generalized second derivative. For $x < y$ the measure is defined by $\mu_f([x, y]) = f'_-(y) - f'_-(x)$. On the set $(0, R_1) \cup (R_1, R_2)$ the measure satisfies

$$\mu_f(dx) = f''(x) dx,$$

since f is twice continuously differentiable. Furthermore,

$$\mu_f((-\infty, 0] \cup (R_2, \infty)) = 0 \quad \text{and} \quad \mu_f(\{R_1, R_2\}) \leq 0.$$

4.1. Proofs of results in Section 1.1.

Proof of Lemma 1. Let $(x, y) \in \mathbb{R}^{2d}$ such that $(x, y) \notin S_2$. Assume w.l.o.g. that $\max\{|x|, |y|\} = |x| \geq \mathcal{R}$. Using our assumption, the triangle inequality and the estimate $\Phi(r) \leq r$, we get

$$V(x) \geq 4C\lambda^{-1} (1 + 2|x|) \geq 4C\lambda^{-1} (1 + |x - y|) \geq 4C\lambda^{-1} (1 + \Phi(|x - y|)).$$

□

Proof of Theorem 1. We use the function f defined at the beginning of Section 4 with the following parameters: The constants R_1 and R_2 are specified by (9) and (13) respectively. For $r \geq 0$ we set $i(r) := 1$, $j(t) := t$ and

$$(68) \quad h(r) := \frac{1}{2} \int_0^r s \kappa(s) ds, \quad \text{where } \kappa \text{ is defined in Assumption 1.}$$

We fix initial values $(x, y) \in \mathbb{R}^{2d}$ and prove (14) for Dirac measures δ_x and δ_y . This is sufficient, since for general $\mu, \nu \in \mathcal{P}_V$ one can show, arguing similarly to [58, Theorem 4.8], that for any coupling γ of μ and ν we have

$$(69) \quad \mathcal{W}_{\rho_1}(\mu p_t, \nu p_t) \leq \int \mathcal{W}_{\rho_1}(\delta_x p_t, \delta_y p_t) \gamma(dx dy).$$

Let $U_t = (X_t, Y_t)$ be a reflection coupling with initial values (x, y) , as defined in Section 3.2. We will argue that $E[e^{ct} \rho_1(X_t, Y_t)] \leq \rho_1(x, y)$ holds for any $t \geq 0$. Denote by $T := \inf \{t \geq 0 : X_t = Y_t\}$ the coupling time. Set $Z_t = X_t - Y_t$ and $r_t = |Z_t|$. The process (Z_t) satisfies the SDE

$$\begin{aligned} dZ_t &= (b(X_t) - b(Y_t)) dt + 2e_t \langle e_t, dB_t \rangle \quad \text{for } t < T, \\ dZ_t &= 0 \quad \text{for } t \geq T, \quad \text{where } e_t = Z_t/r_t. \end{aligned}$$

Until the end of the proof, all Itô equations and differential inequalities hold almost surely for $t < T$, even though we do not mention it every time. An application of Itô's formula shows that (r_t) satisfies the equation

$$dr_t = \langle e_t, b(X_t) - b(Y_t) \rangle dt + 2 \langle e_t, dB_t \rangle \quad \text{for } t < T.$$

Let (L_t^x) denote the right-continuous local time process of the semimartingale (r_t) . Since f is a concave function, we can apply the general Itô-Tanaka formula of Meyer and Wang (cf. e.g. [35, Thm. 22.5] or [49, Ch. VI]) to conclude

$$\begin{aligned} f(r_t) - f(r_0) &= \int_0^t f'_-(r_s) \langle e_s, b(X_s) - b(Y_s) \rangle ds + 2 \int_0^t f'_-(r_s) \langle e_s, dB_s \rangle \\ &\quad + \frac{1}{2} \int_{-\infty}^{\infty} L_t^x \mu_f(dx) \quad \text{for } t < T, \end{aligned} \quad (70)$$

where f'_- denotes the left-derivative of f and μ_f is the non-positive measure representing the second derivative of f , i.e., $\mu_f([x, y]) = f'_-(y) - f'_-(x)$ for $x \leq y$. Moreover, the generalized Itô formula implies for every measurable function $v : \mathbb{R} \rightarrow [0, \infty)$ the equality

$$(71) \quad \int_0^t v(r_s) d[r]_s = \int_{-\infty}^{\infty} v(x) L_t^x dx \quad \text{for any } t < T.$$

Observe that (71) implies that the Lebesgue measure of the set $\{0 \leq s \leq T : r_s \in \{R_1, R_2\}\}$, i.e., the time that (r_s) spends at the points R_1 and R_2 before coupling, is almost surely zero. Our function f is twice continuously differentiable on $(0, \infty) \setminus \{R_1, R_2\}$. The measure $\mu_f(dy)$ is non-positive and thus (71) implies

$$\int_{-\infty}^{\infty} L_t^x \mu_f(dx) \leq \int_{-\infty}^{\infty} I_{\mathbb{R} \setminus \{R_1, R_2\}}(x) f''(x) L_t^x dx = 4 \int_0^t f''(r_s) ds, \quad t < T.$$

We can conclude that a.s. the following differential inequalities hold for $t < T$:

$$\begin{aligned} df(r_t) &\leq (f'(r_t) \langle e_t, b(X_t) - b(Y_t) \rangle + 2 f''(r_t)) dt + 2 f'(r_t) \langle e_t, dB_t \rangle \\ &\leq (-(\beta/2) f(r_t) I_{r_t < R_2} - (\xi/2) I_{r_t < R_1}) dt + 2 f'(r_t) \langle e_t, dB_t \rangle. \end{aligned}$$

For the second inequality, we have used that f is constant on $[R_2, \infty)$ and that inequality (67) holds on $(0, R_2) \setminus \{R_1\}$ with h given by (68). Moreover, we estimated

$$\langle e_t, b(X_t) - b(Y_t) \rangle = \langle Z_t/r_t, b(X_t) - b(Y_t) \rangle \leq \kappa(r_t) r_t, \quad \text{using Assumption 1.}$$

We now turn to the Lyapunov functions. Assumption 2 implies that a.s.

$$d(\epsilon V(X_t) + \epsilon V(Y_t)) \leq 2C\epsilon - \lambda(\epsilon V(X_t) + \epsilon V(Y_t)) dt + dM_t,$$

where (M_t) denotes a local martingale. If $r_t \geq R_1$, the definition of S_1 implies

$$2C\epsilon - \lambda(\epsilon V(X_t) + \epsilon V(Y_t)) \leq -(\lambda/2)(\epsilon V(X_t) + \epsilon V(Y_t)).$$

If $r_t \geq R_2$, then by Assumption 3,

$$2C\epsilon - \lambda(\epsilon V(X_t) + \epsilon V(Y_t)) \leq -(\alpha/2)f(r_t) - (\lambda/2)(\epsilon V(X_t) + \epsilon V(Y_t)),$$

where we have used that by (10) and (65), $\epsilon = \xi/(4C)$ and $\Phi(r) \geq f(r)$. We can conclude that a.s. ,

$$\begin{aligned} d(\epsilon V(X_t) + \epsilon V(Y_t)) &\leq ((\xi/2)I_{r_t < R_1} - (\alpha/2)f(r_t)I_{r_t \geq R_2} - (\lambda/2)(\epsilon V(X_t) + \epsilon V(Y_t))) dt + dM_t. \end{aligned}$$

Summarizing the above results, we can conclude that a.s. , the following differential inequality holds for $t < T$:

$$(72) \quad d\rho_1(X_t, Y_t) = df(r_t) + d(\epsilon V(X_t) + \epsilon V(Y_t)) \leq -c\rho_1(X_t, Y_t) dt + dM'_t,$$

where (M'_t) denotes a local martingale and $c = \min\{\alpha, \beta, \lambda\}/2$. The product rule for semimartingales then implies a.s. for $t < T$:

$$d(e^{ct}\rho_1(X_t, Y_t)) = ce^{ct}\rho_1(X_t, Y_t) dt + e^{ct}d\rho_1(X_t, Y_t) \leq e^{ct}dM'_t.$$

We introduce a sequence of stopping times $(T_n)_{n \in \mathbb{N}}$ given by

$$(73) \quad T_n := \inf\{t \geq 0 : |X_t - Y_t| \leq 1/n \text{ or } \max\{|X_t|, |Y_t|\} \geq n\}.$$

We have $T_n \uparrow T$ a.s. by non-explosiveness. Therefore we finally obtain:

$$\begin{aligned} \mathcal{W}_{\rho_1}(\delta_x p_t, \delta_y p_t) &\leq E[\rho_1(X_t, Y_t)] = E[\rho_1(X_t, Y_t) I_{t < T}] = \lim_{n \rightarrow \infty} E[\rho_1(X_t, Y_t) I_{t < T_n}] \\ &\leq e^{-ct} \liminf_{n \rightarrow \infty} E[e^{c(t \wedge T_n)} \rho_1(X_{t \wedge T_n}, Y_{t \wedge T_n})] \leq e^{-ct} \mathcal{W}_{\rho_1}(\delta_x, \delta_y) \end{aligned}$$

□

Bounds for Example 1. The statement is a special case of Theorem 1. The only thing to verify is the lower bound

$$(74) \quad \beta \geq \sqrt{\kappa/\pi} \left(\int_0^{R_2} \exp(\kappa r^2/4) dr \right)^{-1}.$$

Since $\int_0^\infty \exp(\kappa r^2/4) dr = \sqrt{\kappa/\pi}$, this follows from the definitions of Φ and φ :

$$(75) \quad \beta^{-1} = \int_0^{R_2} \varphi(r)^{-1} \Phi(r) dr \leq \sqrt{\kappa/\pi} \int_0^{R_2} \exp(\kappa r^2/4) dr.$$

□

Proof of Corollary 1. It is well-known that, in our setup, the Markov transition kernels (p_t) admit a unique invariant measure π satisfying $\pi p_t = \pi$ for any $t \geq 0$ and $\int V(x) \pi(dx) \leq C/\lambda$, see e.g. [26]. In [29, Lemma 2.1] it is proven that for any probability measures ν_1 and ν_2 we have

$$\int_{\mathbb{R}^d} V(x) |\nu_1 - \nu_2|(dx) = \inf_{\gamma} \int [V(x) + V(y)] I_{x \neq y} \gamma(dx dy),$$

where the infimum is taken over all couplings γ with marginals ν_1 and ν_2 respectively. In our setup, this implies that for any $\mu \in \mathcal{P}_V$ and $t \geq 0$,

$$\int_{\mathbb{R}^d} V(z) |\mu p_t - \pi|(dz) \leq \epsilon^{-1} \mathcal{W}_{\rho_1}(\mu p_t, \pi p_t) \leq \epsilon^{-1} e^{-ct} \mathcal{W}_{\rho_1}(\mu, \pi).$$

This implies the bound on the mixing time, since

$$\mathcal{W}_{\rho_1}(\delta_x, \pi) \leq \int [f(|x - y|) + \epsilon V(x) + \epsilon V(y)] \pi(dy) \leq R_2 + \epsilon V(x) + \epsilon C/\lambda.$$

□

Proof of Corollary 2. Let $x \in \mathbb{R}^d$. Assumption 2 implies that $\delta_x p_t \in \mathcal{P}_V$ for any $t \geq 0$ and hence $p_t g(x) := E_x[g(X_t)]$ is well defined and finite for any measurable g which is Lipschitz w.r.t. ρ_1 . Fix $(x, y) \in \mathbb{R}^{2d}$ and $t \geq 0$, and let (X_t, Y_t) be an arbitrary coupling of $\delta_x p_t$ and $\delta_y p_t$. We bound the Lipschitz norm of $x \mapsto p_t g(x)$:

$$|p_t g(x) - p_t g(y)| \leq E_{(x,y)}[|g(X_t) - g(Y_t)|] \leq \|g\|_{\text{Lip}(\rho_1)} E_{(x,y)}[\rho_1(X_t, Y_t)].$$

Since the above inequality holds for any such coupling (X_t, Y_t) , Theorem 1 implies

$$\|p_t g\|_{\text{Lip}(\rho_1)} \leq \|g\|_{\text{Lip}(\rho_1)} e^{-ct}.$$

This estimate implies bounds on the bias of ergodic averages:

$$\begin{aligned} & \left| E_x \left[\frac{1}{t} \int_0^t g(X_s) ds - \int g d\pi \right] \right| \leq \frac{1}{t} \int_0^t \int |p_s g(x) - p_s g(y)| \pi(dy) ds \\ & \leq \frac{1 - e^{-ct}}{ct} \|g\|_{\text{Lip}(\rho_1)} \int \rho_1(x, y) \pi(dy) \leq \frac{1 - e^{-ct}}{ct} \|g\|_{\text{Lip}(\rho_1)} \left(R_2 + \epsilon V(x) + \epsilon \frac{C}{\lambda} \right), \end{aligned}$$

where we have used that f is bounded by R_2 .

We now turn to the variance bound. Integrating (20) implies the estimate

$$E_x[V(X_t)^2] \leq C^*/\lambda^* + e^{-\lambda^* t} V(x)^2 \quad \text{for any } t \geq 0.$$

For reals a, b and c the inequality $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$ holds true. Hence

$$\begin{aligned} \int \int \rho_1(y, z)^2 p_t(x, dy) p_t(x, dz) & \leq 3 \left(R_2^2 + 2\epsilon^2 \int V(y)^2 p_t(x, dy) \right) \\ & \leq 3 \left(R_2^2 + 2\epsilon^2 [C^*/\lambda^* + e^{-\lambda^* t} V(x)^2] \right). \end{aligned}$$

Let $A := 3 \left(R_2^2 + 2 \epsilon^2 \left(C^*/\lambda^* + e^{-\lambda^* t} V(x)^2 \right) \right)$. For $t \geq 0$ and $h \geq 0$,

$$\begin{aligned} \text{Var}_x [g(X_t)] &= \frac{1}{2} \int \int (g(y) - g(z))^2 p_t(x, dy) p_t(x, dz) \leq \frac{A}{2} \|g\|_{\text{Lip}(\rho_1)}^2 \quad \text{and} \\ \text{Var}_x [(p_h g)(X_t)] &\leq \frac{A}{2} \|p_h g\|_{\text{Lip}(\rho_1)}^2 \leq \frac{A}{2} \|g\|_{\text{Lip}(\rho_1)}^2 e^{-2ch}. \end{aligned}$$

We get an estimate on the decay of correlations by Cauchy-Schwarz:

$$\begin{aligned} \text{Cov}_x [g(X_t), g(X_{t+h})] &= \text{Cov}_x [g(X_t), (p_h g)(X_t)] \\ &\leq \text{Var}_x [g(X_t)]^{1/2} \text{Var}_x [(p_h g)(X_t)]^{1/2} \leq \frac{A}{2} \|g\|_{\text{Lip}(\rho_1)}^2 e^{-ch}. \end{aligned}$$

Finally, we obtain the variance bound

$$\begin{aligned} \text{Var}_x \left[\frac{1}{t} \int_0^t g(X_s) ds \right] &= \frac{2}{t^2} \int_0^t \int_r^t \text{Cov}_x [g(X_s), g(X_r)] ds dr \\ &= \frac{A}{t^2} \|g\|_{\text{Lip}(\rho_1)}^2 \int_0^t \int_r^t e^{-c(s-r)} ds dr = \frac{A}{ct} \|g\|_{\text{Lip}(\rho_1)}^2. \end{aligned}$$

□

4.2. Proofs of results in Section 1.2.

Proof of Theorem 2. We use the function f defined at the beginning of Section 4 with the following parameters: The constants R_1 and R_2 are specified by (27), we fix $\epsilon \in (0, \infty)$ satisfying (23), set $i(r) := \Phi(r)$ and $j(r) := r$ and define

$$(76) \quad h(r) := \frac{1}{2} \int_0^r s \kappa(s) ds + 2Q(\epsilon)r,$$

where Φ , κ and Q are given by (24), Assumption 1 and (22), respectively.

Fix initial values $(x, y) \in \mathbb{R}^{2d}$. It is enough to prove (28) for Dirac measures δ_x and δ_y , see the proof of Theorem 1 for details. Let $U_t = (X_t, Y_t)$ be a reflection coupling with initial values (x, y) , as defined in Section 3.2. We will argue that $E[e^{ct} \rho_2(X_t, Y_t)] \leq \rho_2(x, y)$ holds for any $t \geq 0$. Denote by $T = \inf \{t \geq 0 : X_t = Y_t\}$ the coupling time. Set $Z_t = X_t - Y_t$ and $r_t = |Z_t|$. The proof of Theorem 1 shows that $f(r_t)$ satisfies a.s. the differential inequality

$$(77) \quad df(r_t) \leq (f'(r_t) \langle e_t, b(X_t) - b(Y_t) \rangle + 2f''(r_t)) dt + 2f'(r_t) \langle e_t, dB_t \rangle$$

for $t < T$, where $e_t = Z_t/r_t$. As in the proof of Theorem 1, the Lebesgue measure of the set $\{0 \leq s \leq T : r_s \in \{R_1, R_2\}\}$, i.e. the time that (r_t) spends at the points R_1 and R_2 before coupling, is almost surely zero. This justifies to write f' and f'' in the above inequality. Observe that Assumption 1 implies the upper bound

$$\langle e_t, b(X_t) - b(Y_t) \rangle = \langle Z_t/r_t, b(X_t) - b(Y_t) \rangle \leq \kappa(r_t) r_t.$$

The function f is constant on $[R_2, \infty)$, and $f(r) \leq \Phi(r)$. Moreover, on $(0, R_2) \setminus \{R_1\}$ the function f satisfies inequality (67). By (77), (67) and (76), we can conclude that a.s. for $t < T$,

$$(78) \quad \begin{aligned} df(r_t) &\leq (-4Q(\epsilon) f'(r_t) - \beta/2 f(r_t) I_{r_t < R_2} - \xi/2 f(r_t) I_{r_t < R_1}) dt \\ &\quad + 2f'(r_t) \langle e_t, dB_t \rangle. \end{aligned}$$

We now turn to the Lyapunov functions and set $G(x, y) := 1 + \epsilon V(x) + \epsilon V(y)$. The definition of the coupling in Section 3.2 implies that a.s. for $t < T$:

$$(79) \quad dG(X_t, Y_t) = (\epsilon \mathcal{L}V(X_t) + \epsilon \mathcal{L}V(Y_t)) dt + \epsilon \langle \nabla V(X_t) + \nabla V(Y_t), dB_t \rangle - 2\epsilon \langle e_t, \nabla V(Y_t) \rangle \langle e_t, dB_t \rangle.$$

Assumption 2 implies $\mathcal{L}V(X_t) + \mathcal{L}V(Y_t) \leq 2C - \lambda(V(X_t) + V(Y_t))$. Notice that by (23), (24) and (65) with $i(r) = \Phi(r)$,

$$(80) \quad 2C\epsilon \leq \left(2 \int_0^{R_1} \Phi(r) \varphi(r)^{-1} dr \right)^{-1} = \xi/2.$$

Recall that $c = \min\{\beta, \lambda, 4C\epsilon\lambda\}/2$. Using the definitions (25) and (26) of the sets S_1 and S_2 respectively, we can conclude that a.s. for $t < T$:

$$(81) \quad d(\epsilon V(X_t) + \epsilon V(Y_t)) \leq (\xi/2 I_{r_t < R_1} - c G(X_t, Y_t) I_{r_t \geq R_2}) dt + \epsilon \langle \nabla V(X_t) + \nabla V(Y_t), dB_t \rangle - 2\epsilon \langle e_t, \nabla V(Y_t) \rangle \langle e_t, dB_t \rangle.$$

By (70) and (79), the covariation of $f(r_t)$ and $\epsilon V(X_t) + \epsilon V(Y_t)$ is, almost surely for $t < T$, given by:

$$d[f(r), \epsilon V(X) + \epsilon V(Y)]_t = 2f'(r_t) \epsilon \langle \nabla V(X_t) - \nabla V(Y_t), e_t \rangle dt.$$

Using Cauchy-Schwarz and (22), we can derive the following bound for any $x, y \in \mathbb{R}^d$ with $x \neq y$:

$$\begin{aligned} \epsilon \left\langle \nabla V(x) - \nabla V(y), \frac{x - y}{|x - y|} \right\rangle &\leq (1 + \epsilon V(x) + \epsilon V(y)) \frac{\epsilon |\nabla V(x)| + \epsilon |\nabla V(y)|}{(1 + \epsilon V(x) + \epsilon V(y))} \\ &\leq 2Q(\epsilon) G(x, y). \end{aligned}$$

Hence, almost surely for $t < T$:

$$(82) \quad d[f(r), \epsilon V(X) + \epsilon V(Y)]_t \leq 4Q(\epsilon) f'(r_t) G(X_t, Y_t) dt.$$

The product rule for semimartingales implies almost surely for $t < T$:

$$d(f(r_t)G(X_t, Y_t)) = G(X_t, Y_t) df(r_t) + f(r_t) dG(X_t, Y_t) + [f(r), G(X, Y)]_t.$$

By (78), we have

$$(83) \quad G(X_t, Y_t) df(r_t) \leq (-\beta/2 \rho_2(X_t, Y_t) I_{r_t < R_2} - \xi/2 \rho_2(X_t, Y_t) I_{r_t < R_1}) dt - 4Q(\epsilon) f'(r_t) G(X_t, Y_t) dt + dM_t^1,$$

where (M_t^1) is a local martingale. Moreover, (81) implies

$$(84) \quad f(r_t) dG(X_t, Y_t) \leq [\xi/2 f(r_t) I_{r_t < R_1} - c \rho_2(X_t, Y_t) I_{r_t \geq R_2}] dt + dM_t^2,$$

where (M_t^2) is again a local martingale. Observe that $G \geq 1$. Combining (82), (83) and (84) we can conclude a.s. for $t < T$:

$$\begin{aligned} d\rho_2(X_t, Y_t) &\leq -c \rho_2(X_t, Y_t) + dM_t, \\ d(e^{ct} \rho_2(X_t, Y_t)) &= c e^{ct} \rho_2(X_t, Y_t) dt + e^{ct} d\rho_2(X_t, Y_t) \leq e^{ct} dM_t, \end{aligned}$$

where (M_t) is a local martingale. We can finish the proof of (28) using a stopping argument, see the end of the proof of Theorem 1 for details. \square

Proof of Corollary 4. Analogously to the proof of Corollary 2, we can conclude that $p_t g(x)$ is finite for any function g which is Lipschitz w.r.t. ρ_2 , any $x \in \mathbb{R}^d$ and $t \geq 0$. Moreover,

$$\|p_t g\|_{\text{Lip}(\rho_2)} \leq \|g\|_{\text{Lip}(\rho_2)} e^{-ct} \quad \text{holds for any } t \geq 0.$$

In particular, for any $x, y \in \mathbb{R}^d$ we can conclude that

$$\begin{aligned} |p_t g(x) - p_t g(y)| &\leq \|g\|_{\text{Lip}(\rho_2)} e^{-ct} \rho_2(x, y) \\ &\leq \|g\|_{\text{Lip}(\rho_2)} e^{-ct} |x - y| (1 + \epsilon V(x) + \epsilon V(y)), \end{aligned}$$

where we used $f(r) \leq r$. If the map $x \mapsto p_t g(x)$ is differentiable at $x \in \mathbb{R}^d$, we can deduce the gradient bound (31). \square

4.3. Proofs of results in Section 1.3.

Proof of Theorem 3. In contrast to the proofs above, we do not use the function f defined in the beginning of Section 4, but the one constructed in [20], i.e., we set

$$f(r) = \int_0^r \varphi(s) g(s \wedge R_2) ds,$$

where φ and g are defined as

$$\varphi(r) = \exp\left(-\frac{1}{2} \int_0^r u \kappa^+(u) du\right) \quad \text{and} \quad g(r) = 1 - \frac{c}{2} \int_0^r \Phi(s) \varphi(s)^{-1} ds.$$

The function Φ and the constant c are given by

$$\Phi(r) = \int_0^r \varphi(s) ds \quad \text{and} \quad c^{-1} = \int_0^{R_2} \Phi(s) \varphi(s)^{-1} ds.$$

The constants R_1 and R_2 are defined in (41) and (42) respectively. Notice that by definition, $\kappa^+(r) = 0$ for any $r \geq R_1$ and thus f is linear on the interval $[R_2, \infty)$. The function f is twice continuously differentiable on $(0, R_2)$, and

$$(85) \quad 2f''(r) = -r \kappa^+(r) f'(r) - c \Phi(r) \leq -r \kappa^+(r) f'(r) - c f(r).$$

We now prove (37) and fix initial values $(x, y) \in \mathbb{R}^{2d}$ as well as a small constant $\delta > 0$. The coupling $U_t := (X_t, Y_t)$, defined in Section 3.3, yields the upper bound

$$\mathcal{W}_{\rho_0}(\mu_t^x, \mu_t^y) \leq E[\rho_0(X_t, Y_t)] = E[f(|X_t - Y_t|)].$$

Let $\gamma := c - |\tau| K$. Set $Z_t := X_t - Y_t$ and $r_t := |Z_t|$. We will argue that there is a constant $C > 0$, independent of δ , such that

$$(86) \quad e^{\gamma t} E[f(r_t)] \leq f(r_0) + e^{\gamma t} C \delta \quad \text{holds true for any } t \geq 0.$$

From this inequality one can then conclude, that for any $t \geq 0$ we have

$$\mathcal{W}_{\rho_0}(\mu_t^x, \mu_t^y) \leq e^{-\gamma t} \rho_0(x, y) + C \delta,$$

which finishes the proof of (37) since $\delta > 0$ can be chosen arbitrarily small. Moreover (38) directly follows from (37) and the inequality

$$r \varphi(R_1) \leq \Phi(r) \leq 2f(r) \leq 2\Phi(r) \leq 2r.$$

We now show (86). By definition of the coupling in Section 3.3,

$$dZ_t = (b^x(t, X_t) - b^y(t, Y_t)) dt + 2 \operatorname{rc}(U_t) e_t dW_t,$$

where $W_t = \int_0^t \langle e_s, dB_s^1 \rangle$ is a one dimensional Brownian motion. Notice that whenever $r_t < \delta/2$, we have $\operatorname{rc}(U_t) = 0$ by definition. Using an approximation argument, cf. [61, Proof of Lemma 3] or arguing similarly to [20, Lemma 6.2], one can show that r_t satisfies almost surely the equation

$$(87) \quad dr_t = \langle \tilde{e}_t, b^x(t, X_t) - b^y(t, Y_t) \rangle dt + 2 \operatorname{rc}(U_t) dW_t,$$

where $\tilde{e}_t := Z_t/r_t$ for $r_t \neq 0$, $\tilde{e}_t := (b^x(t, X_t) - b^y(t, Y_t))/|b^x(t, X_t) - b^y(t, Y_t)|$ if $r_t = 0$ and $|b^x(t, X_t) - b^y(t, Y_t)| > 0$, and \tilde{e}_t is an arbitrary unit vector otherwise. Similarly as in the proof in Section 4.1, we now apply the Itô-Tanaka for semimartingales to conclude that almost surely,

$$\begin{aligned} f(r_t) - f(r_0) &= \int_0^t f'_-(r_s) \langle \tilde{e}_s, b^x(s, X_s) - b^y(s, Y_s) \rangle ds \\ &\quad + 2 \int_0^t \operatorname{rc}(U_s) f'_-(r_s) dW_s + \frac{1}{2} \int_{-\infty}^{\infty} L_t^x \mu_f(dx), \end{aligned}$$

where L_t^x is the right-continuous local time of (r_t) and μ_f is the non-positive measure representing the second derivative of f . By (71), the Lebesgue measure of the set $\{0 \leq s \leq t : r_s \in \{R_1, R_2\}\}$ is almost surely zero. Since f is twice continuously differentiable, except possibly at R_1 and R_2 , we can replace f'_- by f' in the equation above. Moreover, since f is concave, the measure of the points R_1 and R_2 w.r.t. μ_f is non-positive. Hence by (71),

$$\int_{-\infty}^{\infty} L_t^x \mu_f(dx) \leq \int_0^t f''(r_s) d[r]_s = 4 \int_0^t \operatorname{rc}(U_t)^2 f''(r_t) dt \quad \text{a. s.}, \quad \text{and thus}$$

$$(88) \quad f(r_t) = f(r_0) + M_t + \int_0^t H_s ds, \quad \text{where}$$

$$(89) \quad M_t = 2 \int_0^t \operatorname{rc}(U_t) f'(r_t) dW_t, \quad \text{and}$$

$$(90) \quad H_s \leq f'(r_s) \langle \tilde{e}_s, b^x(t, X_s) - b^y(t, Y_s) \rangle + 2 \operatorname{rc}(U_s)^2 f''(r_s).$$

We can bound the inner product using the definitions of b^x , b^y and κ , as well as the Lipschitz bounds on b and ϑ :

$$\begin{aligned} &\langle \tilde{e}_t, b^x(t, X_t) - b^y(t, Y_t) \rangle \\ (91) \quad &= \langle \tilde{e}_t, b(X_t) - b(Y_t) \rangle + \tau \left\langle \tilde{e}_t, \int \vartheta(X_t, z) \mu_t^x(dz) - \int \vartheta(Y_t, z) \mu_t^y(dz) \right\rangle \\ &\leq I_{r_t \geq \delta} r_t \kappa(r_t) + I_{r_t < \delta} \|b\|_{\operatorname{Lip}} \delta + |\tau| L(r_t + \mathcal{W}^1(\mu_t^x, \mu_t^y)). \end{aligned}$$

Notice that $\mathcal{W}^1(\mu_t^x, \mu_t^y) \leq E[r_t]$. Remembering that by (40), $K = 4L/\varphi(R_1)$ and combining (91) with the inequality $r \leq 2f(r)/\varphi(R_1)$, we obtain

$$\begin{aligned} &\langle \tilde{e}_t, b^x(t, X_t) - b^y(t, Y_t) \rangle \\ (92) \quad &\leq I_{r_t \geq \delta} r_t \kappa(r_t) + I_{r_t < \delta} \|b\|_{\operatorname{Lip}} \delta + |\tau| K/2 (f(r_t) + E[f(r_t)]). \end{aligned}$$

The product rule for semimartingales shows that

$$d(e^{\gamma t} f(r_t)) = e^{\gamma t} dM_t + e^{\gamma t} (\gamma f(r_t) + H_t) dt.$$

Using that $\gamma = c - |\tau| K$ and the bound $f' \leq 1$, we can conclude that

$$(93) \quad \begin{aligned} d(e^{\gamma t} f(r_t)) &\leq e^{\gamma t} dM_t + e^{\gamma t} |\tau| K/2 (E[f(r_t)] - f(r_t)) dt \\ &\quad + e^{\gamma t} I_{r_t < \delta} \left(c f(r_t) + \|b\|_{\text{Lip}} \delta \right) dt \\ &\quad + e^{\gamma t} I_{r_t \geq \delta} (c f(r_t) + r_t \kappa(r_t) f'(r_t) + 2f''(r_t)) dt. \end{aligned}$$

Here we used that $f'' \leq 0$ and $\text{rc}(U_t) = 1$ whenever $r_t \geq \delta$. We now argue that for any $r \in (0, \infty) \setminus \{R_2\}$ we have

$$(94) \quad c f(r) + r \kappa(r) f'(r) + 2 f''(r) \leq 0.$$

For $r \in (0, R_2)$ this inequality follows directly from the definition of f , see (85). For $r > R_2$ we have $f''(r) = 0$, but $\kappa(r)$ is sufficiently negative instead: First notice that for $r \geq R_1$, $\varphi(r)$ is constant and hence $\Phi(r) = \Phi(R_1) + \varphi(R_1)(r - R_1)$. Analogously to [20, Theorem 2.2.] we get

$$\begin{aligned} \int_{R_1}^{R_2} \Phi(s) \varphi(s)^{-1} ds &= \int_{R_1}^{R_2} (\Phi(R_1) + \varphi(R_1)(s - R_1)) \varphi(R_1)^{-1} ds \\ &= \Phi(R_1) \varphi(R_1)^{-1} (R_2 - R_1) + (R_2 - R_1)^2/2 \\ &\geq (R_2 - R_1) (\Phi(R_1) + \varphi(R_1)(R_2 - R_1)) \varphi(R_1)^{-1}/2 \\ &= (R_2 - R_1) \Phi(R_2) \varphi(R_1)^{-1}/2. \end{aligned}$$

For $r \geq R_2$ we have $f'(r) = \varphi(R_1)/2$, and thus we get

$$f'(r) r \kappa(r) \leq -2 \frac{\varphi(R_1)}{R_2 - R_1} \frac{r}{R_2} \leq -2 \frac{\varphi(R_1)}{R_2 - R_1} \frac{\Phi(r)}{\Phi(R_2)} \leq -c \Phi(r) \leq -c f(r),$$

where we used the definition of R_2 in (42) and the fact that $c^{-1} = \int_0^{R_2} \Phi(s)/\varphi(s) ds$. Hence (94) holds for any $r \in (0, \infty) \setminus \{R_2\}$. By (93), we conclude that

$$E[e^{\gamma t} f(r_t) - f(r_0)] \leq \delta \left(\|b\|_{\text{Lip}} + c \right) \int_0^t e^{\gamma s} ds,$$

where we used that $f(r) \leq r$. □

We now prepare the proof of Theorem 4 by providing a priori bounds. Notice that Assumption 6 implies that there are constants $A, B > 0$ such that

$$(95) \quad |\vartheta(x, y)| \leq A + B(|x| + |y|) \quad \text{for any } x, y \in \mathbb{R}^d.$$

Lemma 3 (A priori bounds). *Let $V(x) = 1 + |x|^2$. Suppose that Assumptions 6 and 7 hold true. Then there is a constant $C \in (0, \infty)$ such that for any $\tau \in \mathbb{R}$ with $|\tau| \leq \lambda/(8B)$, $x \in \mathbb{R}^d$ and $t \geq 0$, a solution (X_t) of (32) with $X_0 = x$ satisfies*

$$dV(X_t) \leq \left[(C - \lambda V(X_t)) + \left(2|\tau| B |X_t| E[|X_t|] - \frac{\lambda}{4} |X_t|^2 \right) \right] dt + 2 \langle X_t, dB_t \rangle.$$

In particular:

$$E[V(X_t)] \leq C/\lambda + e^{-\lambda t} V(x).$$

Proof of Lemma 3. We apply Itô's formula to $V(X_t)$:

$$(96) \quad \frac{1}{2} dV(X_t) = \langle X_t, b(X_t) \rangle dt + \tau \langle X_t, \int \vartheta(X_t, y) \mu_t^x(dy) \rangle dt + d \, dt + \langle X_t, dB_t \rangle.$$

Using Assumption 7, inequality (95) and $|\tau| \leq \lambda/(8B)$, we conclude

$$\begin{aligned} \frac{1}{2} dV(X_t) &\leq [C_1 - \lambda |X_t|^2 + |\tau| (A |X_t| + B(|X_t|^2 + |X_t| E_x[|X_t|]))] dt + \langle X_t, dB_t \rangle \\ &\leq \left[C_2 - \frac{5}{8} \lambda |X_t|^2 + |\tau| B |X_t| E_x[|X_t|] \right] dt + \langle X_t, dB_t \rangle, \end{aligned}$$

where $C_1 := \sup_{|x| \leq D} |\langle x, b(x) \rangle + \lambda |x|^2 + d|$ and $C_2 > C_1$ is a constant such that

$$-\lambda r^2/4 + |\tau| A r \leq C_2 - C_1 \quad \text{for any } r \in \mathbb{R}_+.$$

It follows that we can find a constant $C > 0$ such that

$$(97) \quad dV(X_t) \leq \left[C - \lambda V(X_t) + 2 |\tau| B |X_t| E[|X_t|] - \frac{\lambda}{4} |X_t|^2 \right] dt + 2 \langle X_t, dB_t \rangle.$$

Applying the product rule for semimartingales we get

$$d(e^{\lambda t} V(X_t)) \leq e^{\lambda t} \left[C + 2 |\tau| B |X_t| E[|X_t|] - \frac{\lambda}{4} |X_t|^2 \right] dt + 2 e^{\lambda t} \langle X_t, dB_t \rangle.$$

We introduce the stopping times $T_n := \inf\{t \geq 0 : |X_t| \geq n\}$ and remark that almost surely, $T_n \uparrow \infty$, since the solution (X_t) is non-explosive. Using Fatou's Lemma and monotone convergence, we can conclude that

$$\begin{aligned} E_x[e^{\lambda t} V(X_t)] &\leq \liminf_{n \rightarrow \infty} E_x[e^{\lambda(t \wedge T_n)} V(X_{t \wedge T_n})] \\ &\leq V(x) + \int_0^t e^{\lambda s} \left[C + 2 |\tau| B E[|X_s|]^2 - \frac{\lambda}{4} E[|X_s|^2] \right] ds. \end{aligned}$$

This concludes the proof, by assumption, $|\tau| \leq \lambda/(8B)$. \square

Proof of Theorem 4. We use the Lyapunov function $V(x) = 1 + |x|^2$. Assumption 7 provides a rate λ and Lemma 3 a constant C . We follow Section 1.2 defining

$$\begin{aligned} S_1 &:= \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : V(x) + V(y) \leq 2C/\lambda\}, \\ S_2 &:= \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : V(x) + V(y) \leq 8C/\lambda\}, \\ R_i &:= \sup\{|x - y| : (x, y) \in S_i\}, \quad i = 1, 2. \end{aligned}$$

We use f defined as in the beginning of Section 4 w.r.t. the following parameters:

$$h(s) := \frac{1}{2} \int_0^s r \kappa(r) dr + 2s, \quad j(s) = s, \quad i(s) := \Phi(s), \quad c := \frac{1}{4} \min\{\beta, \lambda\}, \quad \epsilon := \frac{\xi}{4C},$$

with κ given by Assumption 1. We assume $|\tau| < \lambda/(8B)$ so that Lemma 3 applies.

Fix initial values $(x, y) \in \mathbb{R}^{2d}$, as well as a small constant $\delta > 0$. The coupling $U_t := (X_t, Y_t)$ defined in Section 3.3 yields the upper bound

$$\mathcal{W}_{\rho_2}(\mu_t^x, \mu_t^y) \leq E[\rho_2(X_t, Y_t)].$$

Set $Z_t := X_t - Y_t$ and $r_t := |Z_t|$. Equations (87), (88), (89) and (90) are still valid in our setup. By (91) we can conclude that

$$H_t \leq \left(I_{r_t \geq \delta} f'(r_t) \kappa(r_t) r_t + I_{r_t < \delta} \|b\|_{\text{Lip}} \delta \right) + 2 \text{rc}(U_t)^2 f''(r_t) + |\tau| L(r_t + E[r_t]).$$

By definition, f is constant on $[R_2, \infty)$, and, for $r \in (0, R_2) \setminus \{R_1\}$,

$$2 f''(r) \leq -f'(r) [\kappa(r) r + 4] - (\beta/2) f(r) - (\xi/2) f(r) I_{r < R_1}.$$

Using that f is concave with $f(r) \leq r$ and $\text{rc}(U_t) = 1$ whenever $r_t \geq \delta$, we obtain

$$(98) \quad \begin{aligned} df(r_t) &\leq \left[-\frac{\beta}{2} f(r_t) I_{r_t < R_2} - \frac{\xi}{2} f(r_t) I_{r_t < R_1} - 4 \text{rc}(U_t)^2 f'(r_t) \right] dt \\ &+ \left(\|b\|_{\text{Lip}} + \frac{\beta}{2} + \frac{\xi}{2} \right) \delta dt + |\tau| L(r_t + E[r_t]) dt + 2 \text{rc}(U_t) f'(r_t) dW_t. \end{aligned}$$

Next, we observe that Lemma 3 implies

$$\begin{aligned} dV(X_t) &\leq [C - \lambda V(X_t)] dt + 2 |\tau| B V(X_t) E[V(X_t)] dt + 2 \langle X_t, dB_t \rangle, \\ dV(Y_t) &\leq [C - \lambda V(Y_t)] dt + 2 |\tau| B V(Y_t) E[V(Y_t)] dt + 2 \langle Y_t, d\hat{B}_t \rangle, \end{aligned}$$

where (B_t) and (\hat{B}_t) are the Brownian motions defined in (62). Let

$$G(x, y) := 1 + \epsilon V(x) + \epsilon V(y).$$

The set S_1 is chosen such that

$$2 C \epsilon - \lambda \epsilon V(X_t) + \lambda \epsilon V(Y_t) \leq 0 \quad \text{whenever } r_t \geq R_1.$$

For $r_t \geq R_2$ we have

$$\begin{aligned} 2 C \epsilon - \lambda \epsilon V(X_t) + \lambda \epsilon V(Y_t) &\leq -2 C \epsilon - (\lambda/2) \epsilon V(X_t) - (\lambda/2) \epsilon V(Y_t) \\ &\leq -\min\{\beta/2, \lambda/2\} G(X_t, Y_t), \end{aligned}$$

since $\epsilon = \xi/(4C)$ and $\xi \geq \beta$. We conclude that

$$(99) \quad \begin{aligned} dG(X_t, Y_t) &\leq I_{r_t < R_1} 2 C \epsilon - I_{r_t \geq R_2} \min\{\beta/2, \lambda/2\} G(X_t, Y_t) dt + \epsilon 2 \langle X_t, dB_t \rangle \\ &+ 2\epsilon |\tau| B[V(X_t) E[V(X_t)] + V(Y_t) E[V(Y_t)]] dt + \epsilon 2 \langle Y_t, d\hat{B}_t \rangle. \end{aligned}$$

Note that $|\nabla V(x)| = 2|x| \leq V(x)$. Therefore, and by (88) and (89), we obtain similarly to (82):

$$(100) \quad \begin{aligned} d[f(r), G(X, Y)]_t &= 2 \text{rc}(U_t)^2 f'(r_t) \epsilon \langle \nabla V(X_t) - \nabla V(Y_t), e_t \rangle dt \\ &\leq 2 \text{rc}(U_t)^2 f'(r_t) G(X_t, Y_t) dt. \end{aligned}$$

Using the product rule together with (98), (99) and (100), we see that

$$(101) \quad \begin{aligned} d\rho_2(X_t, Y_t) &= d(f(r_t) G(X_t, Y_t)) \\ &= G(X_t, Y_t) df(r_t) + f(r_t) dG(X_t, Y_t) + d[f(r), G(X, Y)]_t \\ &\leq -\min\{\beta/2, \lambda/2\} f(r_t) G(X_t, Y_t) dt + |\tau| L G(X_t, Y_t) (r_t + E[r_t]) dt \\ &+ 2\epsilon |\tau| B f(r_t) [V(X_t) E[V(X_t)] + V(Y_t) E[V(Y_t)]] dt \\ &+ G(X_t, Y_t) \left(\|b\|_{\text{Lip}} + (\beta + \xi)/2 \right) \delta dt + d\tilde{M}_t, \end{aligned}$$

where (\tilde{M}_t) denotes a local martingale. We further bound the perturbation terms originating from the non-linearity. Notice that for $r < R_2$, inequality (66) holds true and thus there is a constant $K_0 \in (0, \infty)$ s.t. for any $x, y \in \mathbb{R}^d$ we have

$$|x - y| \leq K_0 f(|x - y|) (1 + \epsilon V(x) + \epsilon V(y)) = K_0 \rho_2(x, y).$$

Hence, we can bound

$$|\tau| L G(X_t, Y_t) (r_t + E[r_t]) \leq |\tau| L K_0 (\rho_2(X_t, Y_t) + G(X_t, Y_t) E[\rho_2(X_t, Y_t)]).$$

Moreover,

$$\epsilon V(X_t) E[V(X_t)] + \epsilon V(Y_t) E[V(Y_t)] \leq \epsilon^{-1} E[G(X_t, Y_t)] G(X_t, Y_t).$$

Recall that $2c = \min\{\beta/2, \lambda/2\}$. Hence by the bounds above,

$$d(e^{ct} \rho_2(X_t, Y_t)) = c \rho_2(X_t, Y_t) e^{ct} dt + e^{ct} d\rho_2(X_t, Y_t) \leq e^{ct} J_t dt + e^{ct} d\tilde{M}_t,$$

where

$$\begin{aligned} J_t = & -c \rho_2(X_t, Y_t) + |\tau| L K_0 (\rho_2(X_t, Y_t) + G(X_t, Y_t) E[\rho_2(X_t, Y_t)]) \\ & + 2 |\tau| B \epsilon^{-1} E[G(X_t, Y_t)] \rho_2(X_t, Y_t) + G(X_t, Y_t) \left(\|b\|_{\text{Lip}} + (\beta + \xi)/2 \right) \delta. \end{aligned}$$

Optional stopping and Fatou's lemma now shows that

$$E[e^{ct} \rho_2(X_t, Y_t)] \leq \rho_2(x, y) + \int_0^t e^{cs} E[J_s] ds.$$

Using the a priori bounds from Lemma 3, we see that there is a constant $C_1 \in (0, \infty)$, not depending on δ , such that

$$\left(\|b\|_{\text{Lip}} + (\beta + \xi)/2 \right) \int_0^t e^{cs} E[G(X_s, Y_s)] ds \leq C_1.$$

Since $G \geq 1$, we can conclude that

$$\begin{aligned} & |\tau| L K_0 \int_0^t (E[\rho_2(X_s, Y_s)] + E[G(X_s, Y_s)] E[\rho_2(X_s, Y_s)]) e^{cs} ds \\ & + 2 |\tau| B \epsilon^{-1} \int_0^t E[G(X_s, Y_s)] E[\rho_2(X_s, Y_s)] e^{cs} ds \\ & \leq |\tau| C_2 \int_0^t E[G(X_s, Y_s)] E[\rho_2(X_s, Y_s)] e^{cs} ds, \end{aligned}$$

where $C_2 := 2(L K_0 + B/\epsilon)$. Moreover, the a priori estimates imply

$$\begin{aligned} & \int_0^t e^{cs} E[G(X_s, Y_s)] E[\rho_2(X_s, Y_s)] ds \\ & \leq C_3 \int_0^t e^{cs} E[\rho_2(X_s, Y_s)] ds + C_4(x, y) \int_0^t e^{(c-\lambda)s} E[\rho_2(X_s, Y_s)] ds, \end{aligned}$$

where $C_3 := 1 + \epsilon 2 C/\lambda$ and $C_4(x, y) := \epsilon V(x) + \epsilon V(y)$. If τ is sufficiently small, i.e., if $|\tau| C_2(C_3 + C_4(x, y)) \leq c$, we can conclude that for arbitrary $\delta > 0$,

$$\mathcal{W}_{\rho_2}(\mu_t^x, \mu_t^y) \leq E[\rho_2(X_t, Y_t)] \leq e^{-ct} \rho_2(x_0, y_0) + C_1 \delta.$$

However, observe that C_4 depends on the initial values x and y , i.e. we get a local contraction in the sense that for a given $R > 0$, we can find a constant $\tau_0 \in (0, \infty)$, such that (43) holds for all $|\tau| \leq \tau_0$ and initial values $x, y \in \mathbb{R}^d$ with $|x|, |y| \leq R$. Inequality (44) follows readily from (43) and the definition of K_0 .

In order to obtain a related statement which is valid for any initial condition, see (45), we assume that $|\tau| C_2 C_3 < c$. Similarly as above, we can conclude

$$E[\rho_2(X_t, Y_t)] \leq e^{-ct} \rho_2(x_0, y_0) + C_1 \delta + e^{-ct} |\tau| C_2 C_4(x, y) \int_0^t e^{(c-\lambda)s} E[\rho_2(X_s, Y_s)] ds$$

Using once again the apriori estimates and the bound $f \leq R_2$, we see that

$$\int_0^t e^{(c-\lambda)s} E[\rho_2(X_s, Y_s)] ds \leq R_2(1 + 2\epsilon C/\lambda + \epsilon V(x) + \epsilon V(y)) \int_0^t e^{(c-\lambda)s} ds.$$

Since $\lambda > c$, there is a constant $K_1 \in (0, \infty)$, neither depending on the initial values (x, y) nor on δ , such that

$$E[\rho_2(X_t, Y_t)] \leq e^{-ct} \rho_2(x_0, y_0) + C_1 \delta + e^{-ct} K_1 (\epsilon V(x) + \epsilon V(y))^2.$$

Since $\delta > 0$ is arbitrary, we have shown (45). \square

4.4. Proofs of results in Section 1.4. Before proving Theorem 5, we include a proof of Lemma 2 in Section 2.6. The proof is based on [27, Section 4].

Proof of Lemma 2. The function H is C^2 with strictly positive first derivative, and thus the inverse function $H^{-1} : [0, \infty] \rightarrow [l, \infty]$ is also strictly increasing and C^2 . We define a function $G : [l, \infty) \times [0, \infty) \rightarrow [0, \infty)$ by

$$G(x, t) := H^{-1}(H(x) + ct).$$

Observe that for any fixed $t \geq 0$ the map $x \mapsto G(x, t)$ is a concave C^2 function on (l, ∞) , which can be seen by the following computation:

$$\partial_x^2 G = \partial_x \left(\frac{\eta \circ G}{\eta} \right) = \frac{(\eta \eta') \circ G}{\eta^2} - \frac{(\eta \circ G) \eta'}{\eta^2} \leq 0.$$

Since $x \mapsto G(x, t)$ is concave, Itô's formula shows that almost surely,

$$dG(Z_t, t) \leq \partial_t G(Z_t, t) dt + \partial_x G(Z_t, t) dA_t + dW_t,$$

where (W_t) denotes a local martingale. Observe that $\partial_t G = c \eta \circ G > 0$ and $\partial_x G = \frac{\eta \circ G}{\eta} > 0$. Using our Assumption (57), we can conclude that a.s.

$$dG(Z_t, t) \leq dW_t \quad \text{for } t < T.$$

Let $(T_n)_{n \in \mathbb{N}}$ be a localizing sequence for (W_t) with $T_n \uparrow \infty$ as $n \rightarrow \infty$. We see

$$\begin{aligned} E[G(Z_{t \wedge T}, t \wedge T)] &= E[\liminf_{n \rightarrow \infty} G(Z_{t \wedge T \wedge T_n}, t \wedge T \wedge T_n)] \\ &\leq \liminf_{n \rightarrow \infty} E[G(Z_{t \wedge T \wedge T_n}, t \wedge T \wedge T_n)] \leq E[G(Z_0, 0)] = E[Z_0]. \end{aligned}$$

Since H is non-negative and H^{-1} is increasing, we get

$$E[H^{-1}(c(t \wedge T))] \leq E[G(Z_{t \wedge T}, t \wedge T)] \leq E[Z_0] < \infty.$$

Since the inequality holds for any $t \geq 0$ and $H^{-1}(t) \rightarrow \infty$ as $t \rightarrow \infty$, the stopping time T is almost surely finite, and we can finish the proof using Fatou's lemma:

$$E[H^{-1}(cT)] \leq E[G(Z_T, T)] \leq \liminf_{t \rightarrow \infty} E[G(Z_{t \wedge T}, t \wedge T)] \leq E[Z_0].$$

□

Proof of Theorem 5. We use the function f defined at the beginning of Section 4 with the following parameters: $i \equiv 1$ is the constant function, $j = \eta$, and $h(r) := \frac{1}{2} \int_0^r s \kappa(s) ds$, where κ is defined in Assumption 1.

We now prove (52). Let $U_t = (X_t, Y_t)$ be a reflection coupling with initial values (x, y) , as defined in Section 3.2. Denote by $T := \inf \{t \geq 0 : X_t = Y_t\}$ the coupling time. We will argue that the stochastic process $(\rho_1(X_t, Y_t))$ satisfies the conditions of Lemma 2, except that the map $t \mapsto \rho_1(X_t, Y_t)$ is not continuous at $t = T$. Nevertheless, this obstacle can be overcome by a stopping argument. Set $Z_t = X_t - Y_t$ and $r_t = |Z_t|$. Following the lines of the proof of Theorem 1, one can show that a.s. for $t < T$, $f(r_t)$ satisfies

$$\begin{aligned} df(r_t) &\leq [f'(r_t) \langle e_t, b(X_t) - b(Y_t) \rangle + 2f''(r_t)] dt + 2f'(r_t) \langle e_t, dB_t \rangle \\ &\leq [-\beta/2 \eta(f(r_t)) I_{r_t < R_2} - \xi/2 I_{r_t < R_1}] dt + 2f'(r_t) \langle e_t, dB_t \rangle. \end{aligned}$$

We turn to the Lyapunov functions. Assumption 8 implies that a.s.,

$$d(\epsilon V(X_t) + \epsilon V(Y_t)) \leq (2C\epsilon - (\epsilon \eta(V(X_t)) + \epsilon \eta(V(Y_t)))) dt + dM_t,$$

where (M_t) is a local martingale. Observe that by definition of γ in Theorem 5, and by concavity of η , we have

$$\epsilon(\eta(V(X_t)) + \eta(V(Y_t))) \geq \epsilon \eta(V(X_t) + V(Y_t)) \geq \gamma \eta(\epsilon V(X_t) + \epsilon V(Y_t)).$$

If $r_t \geq R_1$, we know by definition of S_1 that

$$2C\epsilon - (\epsilon \eta(V(X_t)) + \epsilon \eta(V(Y_t))) \leq -\gamma/2 \eta(\epsilon V(X_t) + \epsilon V(Y_t)).$$

If $r_t \geq R_2$, then by Assumption 9,

$$2C\epsilon - (\epsilon V(X_t) + \epsilon V(Y_t)) \leq -\alpha/2 \eta(f(r_t)) - \gamma/2 \eta(\epsilon V(X_t) + \epsilon V(Y_t)),$$

where we have used that η is increasing, $\epsilon = \xi/(4C)$, and $\Phi \geq f$. Thus a.s.,

$$\begin{aligned} d(\epsilon V(X_t) + \epsilon V(Y_t)) &\leq (\xi/2 I_{r_t < R_1} - \alpha/2 \eta(f(r_t)) I_{r_t \geq R_2}) \\ &\quad - \gamma/2 \eta(\epsilon V(X_t) + \epsilon V(Y_t)) dt + dM_t. \end{aligned}$$

Summarizing the above results, we can conclude that almost surely, the following differential inequality holds for $t < T$:

$$\begin{aligned} d\rho_1(X_t, Y_t) &= df(r_t) + d(\epsilon V(X_t) + \epsilon V(Y_t)) \\ (102) \quad &\leq -\min\{\alpha, \beta\}/2 \eta(f(r_t)) - \gamma/2 \eta(\epsilon V(X_t) + \epsilon V(Y_t)) dt + dM'_t \\ &\leq -\min\{\alpha, \beta, \gamma\}/2 \eta(\rho_1(X_t, Y_t)) dt + dM'_t, \end{aligned}$$

where (M'_t) denotes a local martingale and $\min\{\alpha, \beta, \gamma\}/2 = c$.

Now let $T_n := \inf\{t \geq 0 : |X_t - Y_t| \leq \frac{1}{n}\}$. By non-explosiveness we have $T_n \uparrow T$. We have shown that the semimartingale $R_t := \rho_1(X_{t \wedge T_n}, Y_{t \wedge T_n})$ satisfies the assumptions of Lemma 2 for the stopping time T_n . Thus we can conclude

$$\begin{aligned} E[H^{-1}(cT)] &\leq \liminf_{n \rightarrow \infty} E[H^{-1}(cT_n)] \\ &\leq \liminf_{n \rightarrow \infty} E[H^{-1}(H(R_{T_n}) + cT_n)] \leq E[R_0] = \rho_1(x, y). \end{aligned}$$

Inequality (52) now follows from an application of the Markov inequality, and by the fact that H^{-1} is strictly increasing:

$$P[T > t] = P[H^{-1}(cT) > H^{-1}(ct)] \leq \frac{E[H^{-1}(cT)]}{H^{-1}(ct)} \leq \frac{\rho_1(x, y)}{H^{-1}(ct)}.$$

□

Proof of Corollary 5. The proof is similar to the one of [27, Theorem 4.1]. Consider the probability measure $\pi_R(\cdot) := \pi(\cdot \cap A_R)/\pi(A_R)$ where $A_R := \{x \in \mathbb{R}^d : V(x) \leq R\}$ for some constant $R \in (0, \infty)$ to be determined below. Since $\pi p_t = \pi$,

$$\begin{aligned} \|p_t(x, \cdot) - \pi\|_{TV} &\leq \int \|p_t(x, \cdot) - p_t(y, \cdot)\|_{TV} \pi_R(dy) + \|\pi_R p_t - \pi p_t\|_{TV} \\ &\leq \frac{\int \rho_1(x, y) \pi_R(dy)}{H^{-1}(ct)} + \pi(A_R^c) \leq \frac{R_2 + \epsilon V(x) + \epsilon \int V(y) \pi_R(dy)}{H^{-1}(ct)} + \pi(A_R^c), \end{aligned}$$

where we have used that $f \leq R_2$. Similarly to [7, Lemma 4.1], one can see that Assumption 8 implies that the invariant measure π satisfies $\int \eta(V(y)) \pi(dy) \leq C$. Hence, the Markov inequality implies $\pi(A_R^c) \leq C/\eta(R)$. Since $x \mapsto \eta(x)/x$ is non-increasing we have $V(x) \leq \eta(V(x)) R / \eta(R)$ for any $x \in \mathbb{R}^d$ such that $V(x) \leq R$. This yields the upper bound

$$\int_{V \leq R} V d\pi \leq C R / \eta(R).$$

We can conclude that

$$\|p_t(x, \cdot) - \pi\|_{TV} \leq \frac{R_2 + \epsilon V(x)}{H^{-1}(ct)} + \frac{\epsilon C R}{\pi(A_R) \eta(R) H^{-1}(ct)} + \frac{C}{\eta(R)}.$$

We now choose R . Set $b := \eta^{-1}(2C)/l$ and define $R := b H^{-1}(ct)$. Since $\eta(b H^{-1}(0)) = \eta(bl) = 2C$, we also have a lower bound for $\pi(A_R)$:

$$\pi(A_R) = 1 - \pi(A_R^c) \geq 1 - C/\eta(R) \geq 1/2.$$

Combining the bounds, we obtain the assertion

$$\|p_t(x, \cdot) - \pi\|_{TV} \leq \frac{R_2 + \epsilon V(x)}{H^{-1}(ct)} + \frac{C(2\epsilon b + 1)}{\eta(b H^{-1}(ct))}.$$

□

REFERENCES

- [1] Dominique Bakry, Franck Barthe, Patrick Cattiaux, and Arnaud Guillin. A simple proof of the Poincaré inequality for a large class of probability measures including the log-concave case. *Electron. Commun. Probab.*, 13:60–66, 2008.
- [2] Dominique Bakry, Patrick Cattiaux, and Arnaud Guillin. Rate of convergence for ergodic continuous Markov processes: Lyapunov versus Poincaré. *J. Funct. Anal.*, 254(3):727–759, 2008.
- [3] Dominique Bakry, Ivan Gentil, and Michel Ledoux. *Analysis and geometry of Markov diffusion operators*, volume 348 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Cham, 2014.
- [4] François Bolley, Ivan Gentil, and Arnaud Guillin. Convergence to equilibrium in Wasserstein distance for Fokker-Planck equations. *J. Funct. Anal.*, 263(8):2430–2457, 2012.
- [5] François Bolley, Ivan Gentil, and Arnaud Guillin. Uniform convergence to equilibrium for granular media. *Arch. Ration. Mech. Anal.*, 208(2):429–445, 2013.
- [6] Oleg Butkovsky. On ergodic properties of nonlinear Markov chains and stochastic McKean-Vlasov equations. *Theory Probab. Appl.*, 58(4):661–674, 2014.
- [7] Oleg Butkovsky. Subgeometric rates of convergence of Markov processes in the Wasserstein metric. *Ann. Appl. Probab.*, 24(2):526–552, 2014.
- [8] José A. Carrillo, Robert J. McCann, and Cédric Villani. Kinetic equilibration rates for granular media and related equations: entropy dissipation and mass transportation estimates. *Rev. Mat. Iberoamericana*, 19(3):971–1018, 2003.
- [9] José A. Carrillo, Robert J. McCann, and Cédric Villani. Contractions in the 2-Wasserstein length space and thermalization of granular media. *Arch. Ration. Mech. Anal.*, 179(2):217–263, 2006.
- [10] P. Cattiaux, A. Guillin, and F. Malrieu. Probabilistic approach for granular media equations in the non-uniformly convex case. *Probab. Theory Related Fields*, 140(1-2):19–40, 2008.
- [11] Patrick Cattiaux, Arnaud Guillin, Feng-Yu Wang, and Liming Wu. Lyapunov conditions for super poincaré inequalities. *Journal of Functional Analysis*, 256(6):1821–1841, 2009.
- [12] Patrick Cattiaux, Arnaud Guillin, and Li-Ming Wu. A note on Talagrand’s transportation inequality and logarithmic Sobolev inequality. *Probab. Theory Related Fields*, 148(1-2):285–304, 2010.
- [13] Mu Fa Chen and Shao Fu Li. Coupling methods for multidimensional diffusion processes. *Ann. Probab.*, 17(1):151–177, 1989.
- [14] A. S. Dalalyan. Theoretical guarantees for approximate sampling from smooth and log-concave densities. *ArXiv e-prints*, December 2014.
- [15] Randal Douc, Gersende Fort, and Arnaud Guillin. Subgeometric rates of convergence of f -ergodic strong Markov processes. *Stochastic Process. Appl.*, 119(3):897–923, 2009.
- [16] Randal Douc, Gersende Fort, Eric Moulines, and Philippe Soulier. Practical drift conditions for subgeometric rates of convergence. *Ann. Appl. Probab.*, 14(3):1353–1377, 2004.
- [17] A. Durmus and E. Moulines. Non-asymptotic convergence analysis for the Unadjusted Langevin Algorithm. *ArXiv e-prints*, July 2015.
- [18] A. Durmus and E. Moulines. Sampling from strongly log-concave distributions with the Unadjusted Langevin Algorithm. *ArXiv e-prints*, May 2016.
- [19] Andreas Eberle. Reflection coupling and Wasserstein contractivity without convexity. *C. R. Math. Acad. Sci. Paris*, 349(19-20):1101–1104, 2011.
- [20] Andreas Eberle. Reflection couplings and contraction rates for diffusions. *Probability Theory and Related Fields*, pages 1–36. Published online (DOI) 10.1007/s00440–015–0673–1, 2015.
- [21] D. Ferré, L. Hervé, and J. Ledoux. Regular perturbation of V -geometrically ergodic Markov chains. *J. Appl. Probab.*, 50(1):184–194, 2013.
- [22] G. Fort and G. O. Roberts. Subgeometric ergodicity of strong Markov processes. *Ann. Appl. Probab.*, 15(2):1565–1589, 2005.

- [23] M. Hairer, J. C. Mattingly, and M. Scheutzow. Asymptotic coupling and a general form of Harris' theorem with applications to stochastic delay equations. *Probab. Theory Related Fields*, 149(1-2):223–259, 2011.
- [24] M. Hairer, A. M. Stuart, and S. J. Vollmer. Spectral gaps for a Metropolis-Hastings algorithm in infinite dimensions. *Ann. Appl. Probab.*, 24(6):2455–2490, 2014.
- [25] Martin Hairer. Exponential mixing properties of stochastic PDEs through asymptotic coupling. *Probab. Theory Related Fields*, 124(3):345–380, 2002.
- [26] Martin Hairer. Ergodic properties of markov processes. *Lecture notes, Univ. Warwick*. <http://www.hairer.org/notes/Markov.pdf>, 2006.
- [27] Martin Hairer. Convergence of Markov processes. *Lecture notes, Univ. Warwick*. <http://www.hairer.org/notes/Convergence.pdf>, 2016.
- [28] Martin Hairer and Jonathan C. Mattingly. Spectral gaps in Wasserstein distances and the 2D stochastic Navier-Stokes equations. *Ann. Probab.*, 36(6):2050–2091, 2008.
- [29] Martin Hairer and Jonathan C. Mattingly. Yet another look at Harris' ergodic theorem for Markov chains. In *Seminar on Stochastic Analysis, Random Fields and Applications VI*, volume 63 of *Progr. Probab.*, pages 109–117. Birkhäuser/Springer Basel AG, Basel, 2011.
- [30] T. E. Harris. The existence of stationary measures for certain Markov processes. In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. II*, pages 113–124. University of California Press, Berkeley and Los Angeles, 1956.
- [31] R. Z. Has'minskiĭ. Ergodic properties of recurrent diffusion processes and stabilization of the solution of the Cauchy problem for parabolic equations. *Teor. Veroyatnost. i Primenen.*, 5:196–214, 1960.
- [32] Zhenting Hou, Yuanyuan Liu, and Hanjun Zhang. Subgeometric rates of convergence for a class of continuous-time Markov process. *J. Appl. Probab.*, 42(3):698–712, 2005.
- [33] Aldéric Joulin. A new Poisson-type deviation inequality for Markov jump processes with positive Wasserstein curvature. *Bernoulli*, 15(2):532–549, 2009.
- [34] Aldéric Joulin and Yann Ollivier. Curvature, concentration and error estimates for Markov chain Monte Carlo. *Ann. Probab.*, 38(6):2418–2442, 2010.
- [35] Olav Kallenberg. *Foundations of modern probability*. Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2002.
- [36] Rafail Khasminskii. *Stochastic stability of differential equations*, volume 66. Springer, 2011.
- [37] Alexei Kulik. Introduction to ergodic rates for markov chains and processes.
- [38] Torgny Lindvall and L. C. G. Rogers. Coupling of multidimensional diffusions by reflection. *Ann. Probab.*, 14(3):860–872, 1986.
- [39] Mateusz Majka. Coupling and exponential ergodicity for stochastic differential equations driven by Lévy processes. *ArXiv e-prints*, September 2015.
- [40] Florent Malrieu. Convergence to equilibrium for granular media equations and their Euler schemes. *Ann. Appl. Probab.*, 13(2):540–560, 2003.
- [41] MN Malyskin. Subexponential estimates of the rate of convergence to the invariant measure for stochastic differential equations. *Theory of Probability & Its Applications*, 45(3):466–479, 2001.
- [42] J. C. Mattingly. Exponential convergence for the stochastically forced Navier-Stokes equations and other partially dissipative dynamics. *Comm. Math. Phys.*, 230(3):421–462, 2002.
- [43] Sylvie Méléard. Asymptotic behaviour of some interacting particle systems; mckean-vlasov and boltzmann models. In *Probabilistic models for nonlinear partial differential equations*, pages 42–95. Springer, 1996.
- [44] S. P. Meyn and R. L. Tweedie. *Markov chains and stochastic stability*. Communications and Control Engineering Series. Springer-Verlag London, Ltd., London, 1993.
- [45] Sean P. Meyn and R. L. Tweedie. Stability of Markovian processes. I. Criteria for discrete-time chains. *Adv. in Appl. Probab.*, 24(3):542–574, 1992.
- [46] Sean P. Meyn and R. L. Tweedie. Stability of Markovian processes. III. Foster-Lyapunov criteria for continuous-time processes. *Adv. in Appl. Probab.*, 25(3):518–548, 1993.

- [47] Esa Nummelin and Pekka Tuominen. The rate of convergence in Orey's theorem for Harris recurrent Markov chains with applications to renewal theory. *Stochastic Process. Appl.*, 15(3):295–311, 1983.
- [48] N. S. Pillai and A. Smith. Ergodicity of Approximate MCMC Chains with Applications to Large Data Sets. *ArXiv e-prints*, May 2014.
- [49] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*. Springer, Berlin, 2005.
- [50] G. O. Roberts, J. S. Rosenthal, and P. O. Schwartz. Convergence properties of perturbed Markov chains. *J. Appl. Probab.*, 35(1):1–11, 1998.
- [51] Gareth O. Roberts and Jeffrey S. Rosenthal. Quantitative bounds for convergence rates of continuous time Markov processes. *Electron. J. Probab.*, 1:no. 9, approx. 21 pp. (electronic), 1996.
- [52] D. Rudolf and N. Schweizer. Perturbation theory for Markov chains via Wasserstein distance. *ArXiv e-prints*, March 2015.
- [53] T. Shardlow and A. M. Stuart. A perturbation theory for ergodic Markov chains and application to numerical approximations. *SIAM J. Numer. Anal.*, 37(4):1120–1137, 2000.
- [54] Alain-Sol Sznitman. Topics in propagation of chaos. In *École d'Été de Probabilités de Saint-Flour XIX—1989*, volume 1464 of *Lecture Notes in Math.*, pages 165–251. Springer, Berlin, 1991.
- [55] Pekka Tuominen and Richard L. Tweedie. Subgeometric rates of convergence of f -ergodic Markov chains. *Adv. in Appl. Probab.*, 26(3):775–798, 1994.
- [56] A. Yu. Veretennikov. On polynomial mixing bounds for stochastic differential equations. *Stochastic Process. Appl.*, 70(1):115–127, 1997.
- [57] Cédric Villani. *Optimal transport*, volume 338 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2009. Old and new.
- [58] Max-K. von Renesse and Karl-Theodor Sturm. Transport inequalities, gradient estimates, entropy, and Ricci curvature. *Comm. Pure Appl. Math.*, 58(7):923–940, 2005.
- [59] Jian Wang. L^p -Wasserstein distance for stochastic differential equations driven by Lévy processes. *Bernoulli*, 22(3):1598–1616, 2016.
- [60] Raphael Zimmer. Kantorovich contractions of diffusion semigroups with explicit constants. *PhD Thesis, University of Bonn, in preparation*.
- [61] Raphael Zimmer. Explicit contraction rates for a class of degenerate and infinite-dimensional diffusions. *ArXiv e-prints*, May 2016.

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